Projections of Gibbs measures on self-conformal sets

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Self-similarity

Let $\mathcal{I}$ be an iterated function system (IFS) of the following form

$$\mathcal{I} = \{f_i \cdot = r_i O_i \cdot + t_i\}_{i=1}^{m}$$

where $0 < r_i < 1$, $O_i \in SO(d)$, $t_i \in \mathbb{R}$. It is well known that there exists a unique non-empty compact set X such that $X = \cup_{i=1}^{m} f_i(X)$. We call this X a *self-similar set*, on which we define a *self-similar measure* μ. We say that $\mathcal{I}$ has *dense rotations* if the rotation group $G = \langle O_i \rangle$ is dense in $SO(d)$. We denote by $\dim_H$ the Hausdorff dimension, and by $\Pi_{d,k}$ the set of orthogonal projections from $\mathbb{R}^d$ to its k-dimensional subspaces.

Strong Marstrand result for self-similar measures

Theorem 1: (Hochman & Shmerkin, [3])
For self-similar μ, with dense rotations and satisfying the strong separation condition, we have

$$\dim_H \pi\mu = \min\{k, \dim_H \mu\} \tag{1}$$

for all $\pi \in \Pi_{d,k}$.

This theorem implies a strengthening of Marstrand's famous projection theorem [4] for self-similar sets.

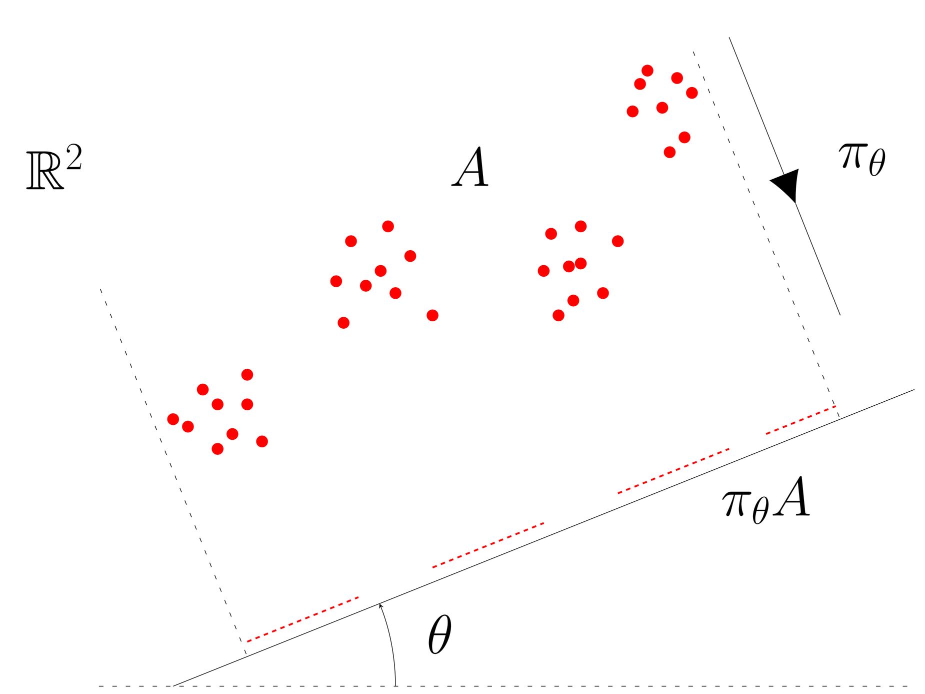

Figure: Marstrand's Projection Theorem: for almost every π_θ with respect to the natural Haar measure, $\dim_H \pi_\theta A = \min\{\dim_H A, 1\}$. This holds for any A Borel or analytic.

CP-chains

The proof of Theorem 1 utilises Furstenberg's CP-chain method, which involves studying the dynamics of a sequence of *zoom-in measures*, which we define in the following way

$$\mu_A = \frac{\mu|_A}{\mu(A)}. \tag{2}$$

Geometric properties of these measures can be used to deduce information about the original measure.

Self-conformal measures

We now consider a more general class of IFSs. Let $\Lambda = \{1, ..., m\}$ and let

$$\mathcal{I} = \{f_i\}_{i \in \Lambda},$$

where each f_i is *conformal* on an open set U, i.e. $f_i'(x) = r_i(x)O_i(x)$ with $0 < r_i(x) < 1$, $O_i(x) \in SO(d)$. We call the attractor K a *self-conformal set* on which we consider a Gibbs measure for a Hölder potential.

Symbolic space

We consider the space of infinite sequences $\Lambda^{\mathbb{N}}$. For $i = i_1 i_2 \cdots \in \Lambda^{\mathbb{N}}$ and $n \geq 1$ denote by $i|_n = i_1 \cdots i_n$, and $f_{i|_n} = f_{i_1} \circ \cdots \circ f_{i_n}$. Fix $x_0 \in U$ and let $\Phi : \Lambda^{\mathbb{N}} \to K$ be the canonical mapping

$$\Phi(i) = \lim_{n\to\infty} f_{i|_n}(x_0). \tag{3}$$

This is a natural symbolic representation of K.

Dense rotations assumption

In order to maintain invariance while zooming in on our measure we consider the product space $\Lambda^{\mathbb{N}} \times SO(d)$, where $SO(d)$ is the rotation group of our IFS, along with a skew product. To ensure this product dynamical system is ergodic we assume

(A) the skew product has a dense orbit in $\Lambda^{\mathbb{N}} \times SO(d)$.

This can be thought of as an analogue of the dense rotations assumption in the self-similar case.

Main result

Under (A), for μ a Gibbs measure on a self-conformal set K, we have

$$\dim_H \pi\mu = \min\{k, \dim_H \mu\} \tag{4}$$

for all $\pi \in \Pi_{d,k}$.

Remark

This result requires no separation condition. This is due to first viewing the system in symbolic space and then applying the canonical projection (3)*, a method from [2].*

Idea of proof

We were able to follow the same general method of proof as with self-similar measures. The main difficulty is that in a conformal IFS the contraction ratio and rotation vary from point to point. This is a problem when trying to estimate entropy of the zoom-in measures. We overcome this using the fact that at very small scales, Gibbs measures on conformal systems start to look more and more like Bernoulli measures on self-similar systems. Thus we can control the error between our measure and a self-similar measure, which enables us to make the required estimates for entropy.

Julia sets

The most common example of a self-conformal set is a Julia set. For a Julia set J defined by the complex polynomial $f(z) = z^2 + c$ to satisfy assumption (A), the following is sufficient. There exists a periodic point $z \in J$ i.e. $f^n(z) = z$ for some $n \geq 1$ such that

$$\frac{\arg\{(f^n)'(z)\}}{\pi} \notin \mathbb{Q}.$$

Since periodic points are dense in the Julia set, we believe this to be a reasonable assumption. We are interested in classifying precisely which Julia sets meet this condition.

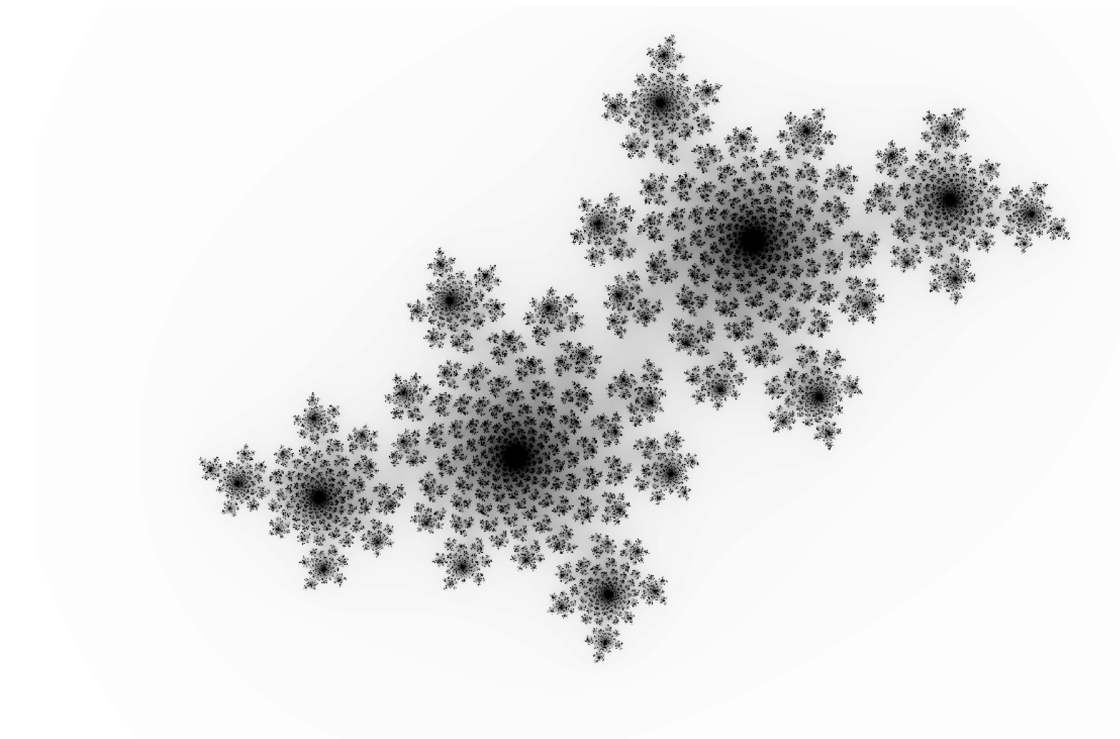

Figure: An example of a self-conformal Julia set. Julia sets defined by quadratics of the form $f(z) = z^2 + c$ can be viewed as attractors of the IFS consisting of the two inverse branches of f.

References

[1] C. Bruce and X. Jin, *Projections of Gibbs measures on self-conformal sets.* arXiv preprint arXiv:1801.06468, 2018.

[2] K. Falconer and X. Jin, *Exact dimensionality and projections of random self-similar measures and sets*, J. London Math. Soc. (2) 90: 388-412, 2014.

[3] M. Hochman and P. Shmerkin, *Local entropy averages and projections of fractal measures*, Ann. of Math. (2) 175: 1001-1059, 2012.

[4] J. M. Marstrand, *Some fundamental geometrical properties of plane sets of fractional dimensions*, Proc. London Math. Soc. 3.1: 257-302, 1954.