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Good labelling property of simple nested fractals

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Motivation

K. Pietruska-Pałuba in [1] constructed the reflected Brownian motion on the Sierpiński gasket. A crucial element of the construction was the labelling of the fractal vertices which allowed to define folding projections on a complex of a given size.
The construction of the reflected Brownian motion is a key step in proving the existence of the Integrated density of states (IDS) for the subordinate Brownian motions perturbed by random potentials.
In [2] the reflected Brownian motion was constructed on simple nested fractals satisfying the Good labelling property (GLP).

Notation

Let $\Psi_i : \mathbb{R}^2 \to \mathbb{R}^2, 1 \le i \le N$ be similitudes given by formula

$$\Psi_i(x) = (1/L)U(x) + \nu_i,$$

where U is an isometry of $\mathbb{R}^2$, $L > 1$ is a scaling factor, $\nu_i \in \mathbb{R}^2$ (for future calculations we shall assume that $\nu_1 = 0$).
There exists the unique nonempty compact set $\mathcal{K}^{\langle 0\rangle}$ such that

$$\mathcal{K}^{\langle 0\rangle} = \bigcup_{i=1}^{N} \Psi_i\left(\mathcal{K}^{\langle 0\rangle}\right).$$

The set $\mathcal{K}^{\langle 0\rangle}$ is the (bounded) fractal.

We shall denote $\mathcal{K}^{\langle M\rangle} = L^M \mathcal{K}^{\langle 0\rangle}$.
Every set $\Delta_M \subset \mathcal{K}^{\langle\infty\rangle}$ of the form

$$\Delta_M = \mathcal{K}^{\langle M\rangle} + \nu_{\Delta_M},$$

where $\nu_{\Delta_M} = \sum_{j=M+1}^{J} L^j \nu_{i_j}$, for some $J \ge M+1$, $\nu_{i_j} \in \{\nu_1, ..., \nu_N\}$, is called an M-complex.

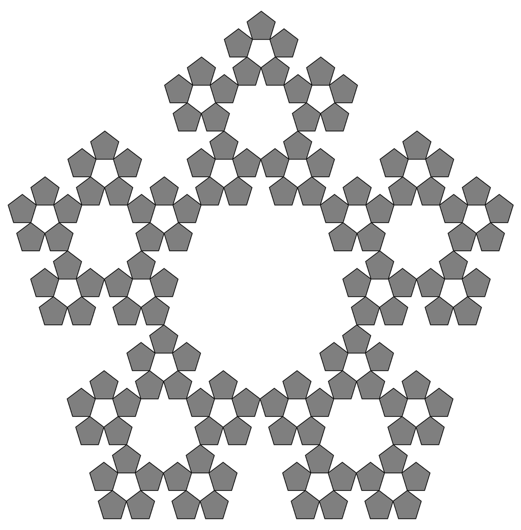

Figure: The Sierpiński Pentagon: 5 similitudes, $L = \frac{3+\sqrt{5}}{2}$

Essential fixed points

The fixed point $x \in \mathcal{K}^{\langle 0\rangle}$ is an *essential fixed point* if there exists another fixed point $y \in \mathcal{K}^{\langle 0\rangle}$ and similitudes Ψ_i, Ψ_j such that $\Psi_i(x) = \Psi_j(y)$.
$V_0^{\langle 0\rangle}$ is the set of the essential fixed points, $\#V_0^{\langle 0\rangle} = k$.
We also denote $V_M^{\langle M\rangle} = L^M V_0^{\langle 0\rangle}$.

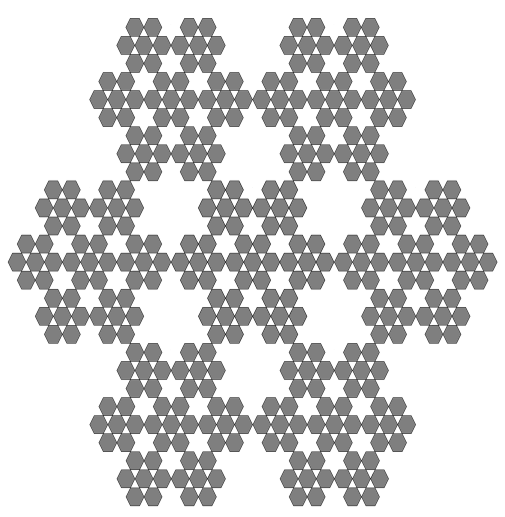

Figure: The Lindstrøm snowflake: 7 similitudes, $L = 3$ (7 fixed points, but only 6 essential fixed points).

Unbounded simple nested fractals (USNF)

The set $\mathcal{K}^{\langle\infty\rangle} = \bigcup_{M=0}^{\infty} L^M \mathcal{K}^{\langle 0\rangle}$ is called an *unbounded simple nested fractal (USNF)* if the following conditions regarding the family of similitudes Ψ_i generating the compact fractal $\mathcal{K}^{\langle 0\rangle}$ and the set of vertices $V_0^{\langle 0\rangle}$ meet.

1. $\#V_0^{\langle 0\rangle} \ge 2$.
2. There exists an open set $U \in \mathbb{R}^2$ such that for $i \ne j$ $\Psi_i(U) \cap \Psi_j(U) = \emptyset$ and $\bigcup_i \Psi_i(U) \subseteq U$.
3. (Nesting) Let T, S be different 1-complexes. Then $T \cap S = V(T) \cap V(S)$.
4. (Symmetry) For $x, y \in V_0^{\langle 0\rangle}$ let $R_{x,y}$ denote the symmetry with respect to hyperplane bisecting the segment $[x, y]$. Then
$$\forall i \in \{1, ..., N\}\ \forall x, y \in V_0^{\langle 0\rangle}\ \exists j \in \{1, ..., N\}$$
$$R_{x,y}\left(\Psi_i\left(V_0^{\langle 0\rangle}\right)\right) = \Psi_j\left(V_0^{\langle 0\rangle}\right)$$
5. (Connectivity) On the set $V_{-1}^{\langle 0\rangle} = \bigcup_i \Psi_i(V_0^{\langle 0\rangle})$ we define graph structure E_{-1} as follows:
$(x, y) \in E_{-1}$ if x and y are in the same -1-complex.
Then the graph $(V_{-1}^{\langle 0\rangle}, E_{-1})$ is connected.

Shape of the complexes.

Proposition. If $k = 2$, then the fractal is just a segment connecting two essential fixed points.
If $k \ge 3$, then the points from $V_0^{\langle 0\rangle}$ are vertices of a regular polygon.

Definition of the good labelling property (GLP)

Since vertices of every M-complex Δ_M are the vertices of a regular k-gon, there exist exactly k different rotations around the barycenter of $\mathcal{K}^{\langle M\rangle}$, mapping $V_M^{\langle M\rangle}$ onto $V_M^{\langle M\rangle}$. They will be denoted by $\mathcal{R}_M := \{R_1, ..., R_k\}$ (ordered in such a way that for $i = 1, 2, ..., k$, the rotation R_i rotates by angle $\frac{2\pi i}{k}$).
Let us consider the set of k letters $\mathcal{A} := \{a_1, a_2, a_3, ..., a_k\}$, called labels.
Definition. Let $M \in \mathbb{Z}$. A function $l_M : V_M^{\langle\infty\rangle} \to \mathcal{A}$ is called *a good labelling function of order M* if the following conditions meet.
(1) The restriction of l_M to $V_M^{\langle M\rangle}$ is a bijection onto $\mathcal{A}$.
(2) For every M-complex Δ_M represented as
$$\Delta_M = \mathcal{K}^{\langle M\rangle} + \nu_{\Delta_M},$$
there exists a rotation $R_{\Delta_M} \in \mathcal{R}_M$ such that
$$l_M(v) = l_M\left(R_{\Delta_M}\left(v - \nu_{\Delta_M}\right)\right), \quad v \in V\left(\Delta_M\right).$$

An USNF $\mathcal{K}^{\langle\infty\rangle}$ is said to have the *good labelling property of order M* if a good labelling function of order M exists. Note that, in fact, for every M-complex Δ_M the restriction of a good labelling function to $V\left(\Delta_M\right)$ is a bijection onto $\mathcal{A}$.
Remark. Thanks to the self-similar structure of the set $\mathcal{K}^{\langle\infty\rangle}$ it has a *good labelling property of order M* for some $M \in \mathbb{Z}$ if and only if it has this property for every $M \in \mathbb{Z}$. We can simply say that $\mathcal{K}^{\langle\infty\rangle}$ has a *good labelling property (GLP)*.

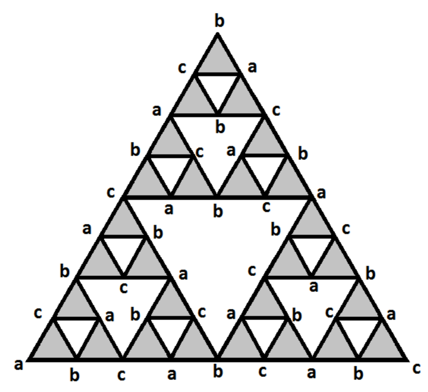

Figure: Labelling of vertices of the Sierpiński gasket.

Definition of the projection

We define a projection map π_M from the unbounded fractal $\mathcal{K}^{\langle\infty\rangle}$ onto the primary M-complex $\mathcal{K}^{\langle M\rangle}$ by setting

$$\pi_M(x) = R_{\Delta_M}\left(x - \nu_{\Delta_M}\right)$$

where $\Delta_M = \mathcal{K}^{\langle\infty\rangle} + \nu_{\Delta_M}$ is the M-complex containing x.

Reflected process

The reflected Brownian motion on $\mathcal{K}^{\langle M\rangle}$ is defined as follows:

$$X_t^M = \pi_M\left(Z_t\right),$$

where Z_t is a diffusion on unbounded fractal $\mathcal{K}^{\langle\infty\rangle}$.

Impossibility of labelling the Lindstrøm snowflake

Can we consistently label vertices of any nested fractal?
No, there exist fractals which cannot be labelled, e.g. the Lindstrøm snowflake.
Having labelled vertices of the bottom left complex clockwise as a, b, c, d, e, f we know that the bottom right complex must have its left vertex labelled as c. Labelling other vertices of this complex clockwise determines that the label of the top left vertex is d. On the other hand, the middle complex has the bottom left vertex labelled as b, therefore its bottom right vertex should be labelled as a. The vertex cannot have two labels, therefore this fractal cannot be well labelled.

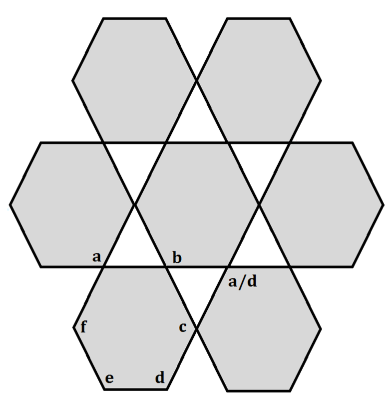

Figure: Illegal labelling of vertices of the Lindstrøm snowflake.

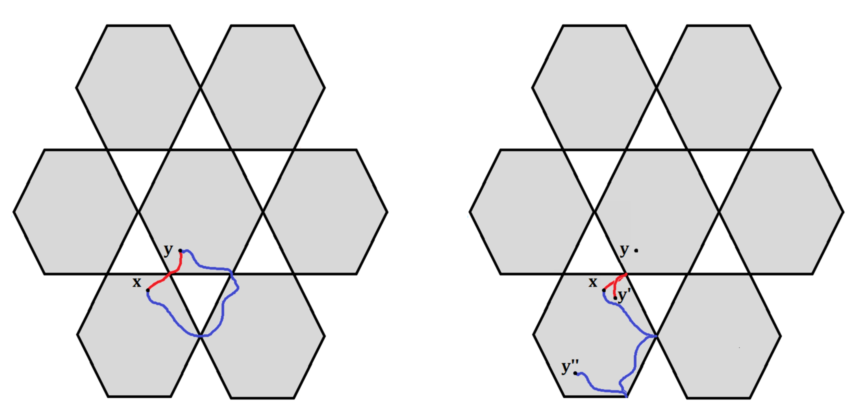

Figure: The endpoints of the reflected trajectories on snowflake depend on whole past of the process - the constructed process on snowflake would not have the Markov property.

Easy-to-check condition for GLP

The fractal $\mathcal{K}^{\langle\infty\rangle}$ has the good labelling property if and only if there exists a function $\widetilde{\ell}_0 : V_0^{\langle 1\rangle} \to \mathcal{A}$ such that the restriction of $\widetilde{\ell}_0$ to $V_0^{\langle 0\rangle}$ is a bijection into $\mathcal{A}$ and the condition (2) is satisfied for every 0-complex $\Delta_0 \subset \mathcal{K}^{\langle 1\rangle}$.

GLP Theorem 1.

If $k = \#V_0^{\langle 0\rangle}$ is prime, then $\mathcal{K}^{\langle\infty\rangle}$ has the good labelling property.

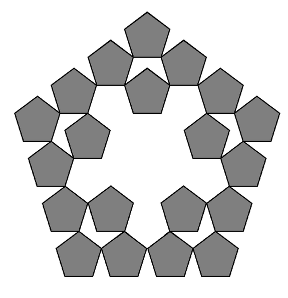

Figure: An example of a fractal with five essential fixed points (description of one iteration).

GLP Theorem 2.

If $k = \#V_0^{\langle 0\rangle} = N \ge 3$, then $\mathcal{K}^{\langle\infty\rangle}$ has the good labelling property.

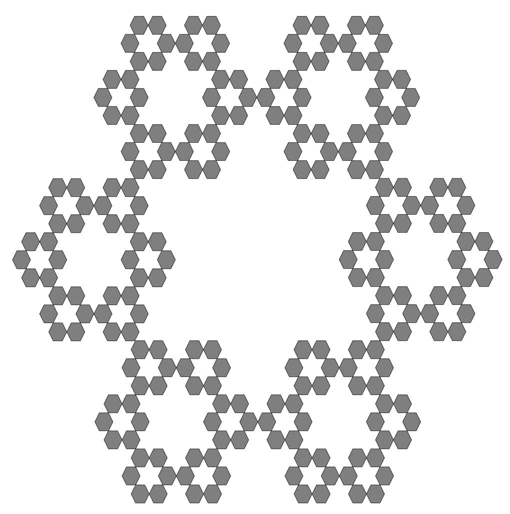

Figure: An example of a fractal with six essential fixed points and six fixed points - complexes create ring structures (after three iterations).

GLP Theorem 3.

If $2|k = \#V_0^{\langle 0\rangle}$, $k > 2$, then $\mathcal{K}^{\langle\infty\rangle}$ has the GLP if and only if the 0-complexes inside the 1-complex $\mathcal{K}^{\langle 1\rangle}$ can be divided into two classes such that each complex from one class intersects only with the complexes from the other class.
Remark: If $k = 4$, then the condition above is always satisfied and $\mathcal{K}^{\langle\infty\rangle}$ has the GLP.

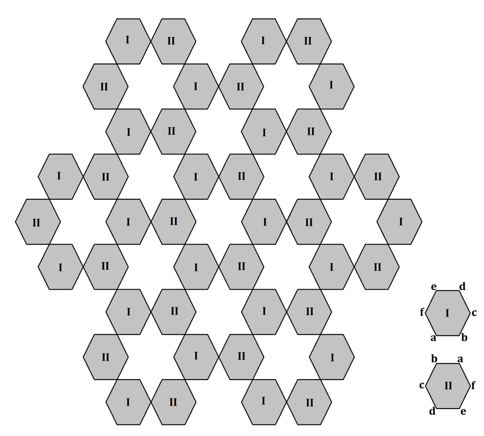

Figure: An example of a fractal with six essential fixed points, $N = 42$ (description of one iteration). The complexes are divided into two classes.

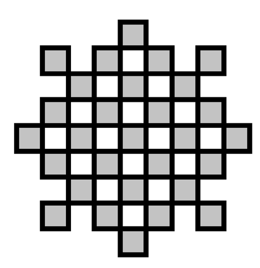

Figure: An example of a fractal with four essential fixed points (description of one iteration).

Summary

The fractals not considered in any of the theorems above are the fractals for which k is odd, composite and $N > k$. Below there is an example of such fractal that cannot be well-labelled.

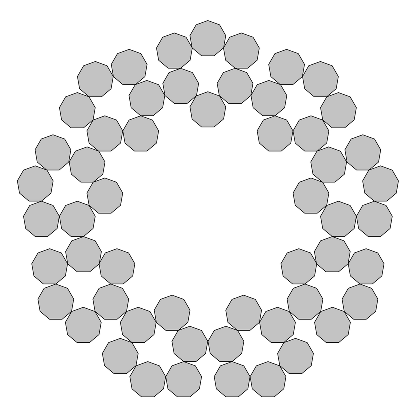

Figure: An example of a fractal with nine essential fixed points, $N = 54$, without the GLP (description of one iteration).

Bibliography

[1]: K. Pietruska-Pałuba *The Lifschitz singularity for the density of states on the Sierpinski gasket*, Probab. Theory Related Fields 89 (1991), no. 1, 1-33.
[2]: K. Kaleta, M. Olszewski, K. Pietruska-Pałuba *Reflected Brownian motion on simple nested fractals*, arXiv preprint 1804.04228 (2018).