Walk dimension of Vicsek sets

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Introduction

Our main goal is to determine the walk dimension $\dim_W$ of generalized Vicsek sets with scaling ratios $r = (2n+1)^{-1}, n \in \mathbb{N}$ without using diffusion. As an example the classical Vicsek set with scaling ratio $r = 1/3$ is shown in Figure 1.

For any $n \in \mathbb{N}$ the construction of the Vicsek sets follows the same pattern. Starting with the unit square we scale it by the factor $r = (2n+1)^{-1}$ and arrange $m = 4n+1$ copies of it on the diagonals of the unit square. We then scale the whole set again by r and arrange m copies on the diagonals of the unit square as depicted in Figure 2.

Boundaries of augmented trees

Consider a sequence of graphs instead of sets when constructing self similar fractals. In the case of the Vicsek sets we start with the complete graph with four nodes K_4. We then take m rescaled copies of K_4 and connect them as shown in Figure 3. Continuing this leaves us with a sequence of graphs $(V_k)_{k\geq 0}$ which converges to the Vicsek set. By connecting them in an ascending order with vertical edges (red) and adding horizontal edges (blue) as in Figure 4 we get an augmented tree $X = (V, E)$. In [LW09] Lau and Wang showed that this augmented tree is hyperbolic if the IFS satisfies the OSC. Since this is the case for the Vicsek sets, our augmented tree is hyperbolic. Also the self similar set K of the IFS is Hölder equivalent to the hyperbolic boundary $\partial_H X$ of X.

We now define a so called λ-return ratio random walk which is introduced in [KLW17]. This random walk has the property

$$\frac{a_{x,x^-}}{\sum_{y:y^-=x} a_{x,y}} = \frac{p_{x,x^-}}{\sum_{y:y^-=x} p_{x,y}} = \lambda, \qquad (1)$$

where x^- is the parent node of x in X. By choosing the same value for all conductances on vertical edges

$$a_{x,y} = a_{k,k+1},\ x \in V_k, y \in V_{k+1}, k \geq 0,$$

equation (1) simplifies to

$$\lambda = \frac{a_{k-1,k}}{(4n+1)a_{k,k+1}},$$

and by recursion this yields

$$a_{k,k+1} = \frac{1}{((4n+1)\lambda)^k}$$

if $a_{0,1} = 1$. The conductances on the horizontal edges are given by

$$a_{x,y} = \begin{cases} \frac{C_1}{((4n+1)\lambda)^k} & \text{type I (black),} \\ \frac{C_2}{((4n+1)\lambda)^k} & \text{type II (blue).} \end{cases}$$

By [KLW17] this random walk is transient and we can define the Martin boundary $\mathcal{M}$ of X. Additionaly the the Martin boundary $\mathcal{M}$, the hyperbolic boundary $\partial_H X$ and the self similar set K are homeomorphic.

Energy on Graphs

For a graph (F, E, A) we define the energy on the graph by

$$\mathcal{E}_F(u,u) = \frac{1}{2}\sum_{x,y\in X} a_{x,y}(u(x)-u(y))^2.$$

It is easy to see that

$$\mathcal{E}_F(u,u) = -u^T A u$$

where A is the conductance matrix of F, with

$$a_{x,y} \geq 0,\ x \neq y;\ a_{x,y} = a_{y,x};\ a_{x,x} = -\sum_{y\neq x} a_{x,y}$$

and $u = (u(x_1), \dots, u(x_l))^T, x_1, \dots, x_l \in F$.

We then can determine the trace energy on a subset of F. Let $G \subset F$ and $H = F \setminus G$. By rearranging A we can write it as follows

$$A = \left(\begin{array}{c|c} A_{GG} & A_{GH} \\ \hline A_{HG} & A_{HH} \end{array}\right).$$

In [Bar98] Barlow showed that the trace energy on G is then defined by the Schur complement

$$B = A_{GG} - A_{GH}A_{HH}^{-1}A_{HG}. \qquad (2)$$

Let P be the transition matrix of F where all nodes in G are replaced by absorbing states. P then has the form

$$P = \left(\begin{array}{c|c} I_{|G|} & 0 \\ \hline Q & R \end{array}\right).$$

For a connected graph F any of the absorbing states will be reached eventually by the Markov chain X corresponding to P. Hence the following limit exists:

$$\tilde{P} := \lim_{n\to\infty} P^n = \lim_{n\to\infty} \left(\begin{array}{c|c} I_{|G|} & 0 \\ \sum_{i=0}^n R^i Q & 0 \end{array}\right) = \left(\begin{array}{c|c} I_{|G|} & 0 \\ (I_{|H|} - R)^{-1}Q & 0 \end{array}\right).$$

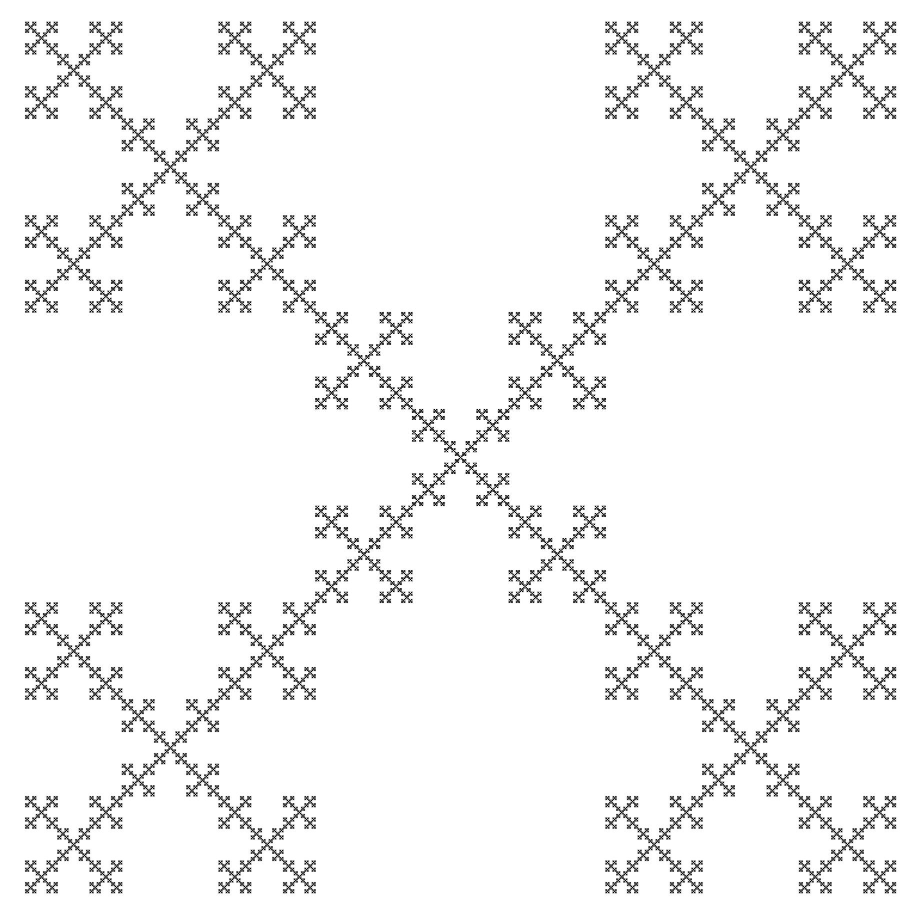

Figure 1: Vicsek set with scaling factor $\frac{1}{3}$.

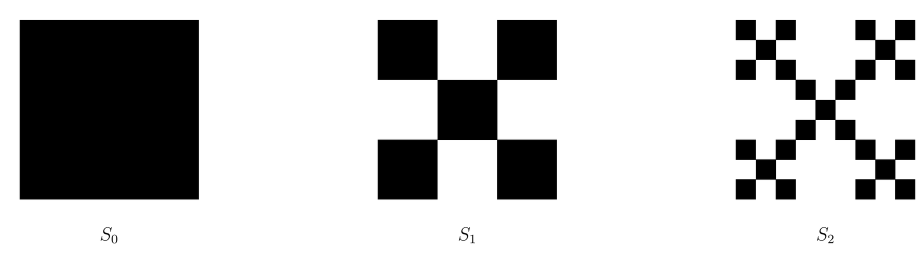

Figure 2: The first two steps of the construction of the Vicsek set.

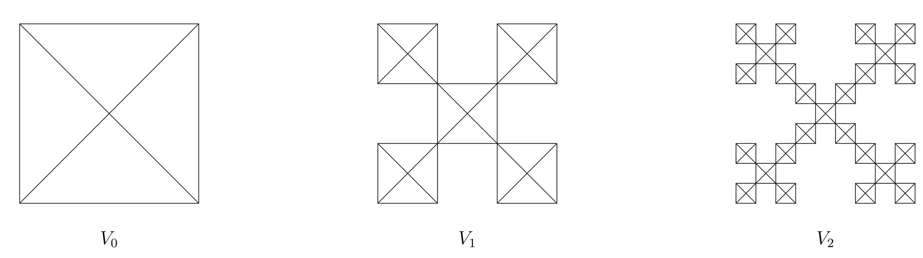

Figure 3: The first two steps of the construction of the Vicsek set with graphs.

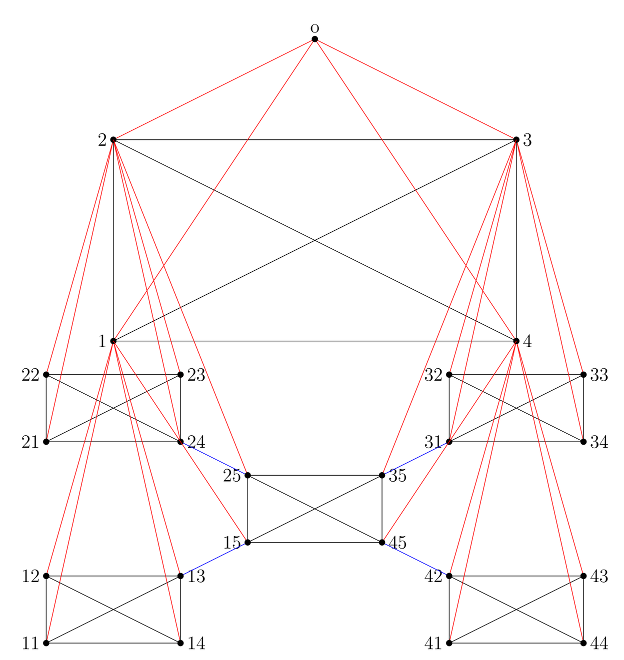

Figure 4: The first three layers of the augmented tree for the Vicsek set.

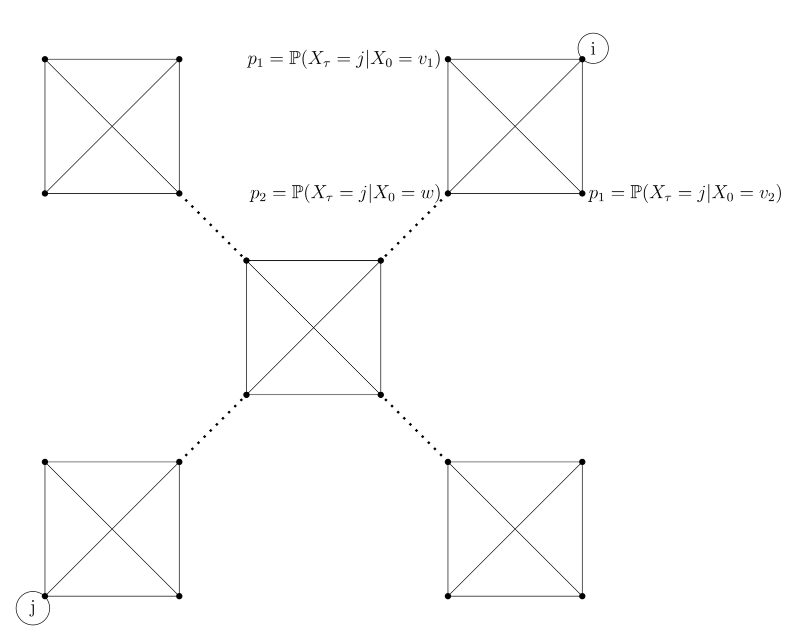

Figure 5: Probabilities on a pregraph of the Vicsek set.

References

[Bar98] Martin T. Barlow. *Lectures on probability theory and statistics*, chapter Diffusions on fractals, pages 1–121. Springer, 1998.

[GY17] A. Grigor'yan and M. Yang. Determination of the Walk Dimension of the Sierpiński Gasket Without Using Diffusion. 2017.

[KLW17] Shi-Lei Kong, Ka-Sing Lau, and Ting-Kam Leonard Wong. Random walks and induced dirichlet forms on self-similar sets. *Advances in Mathematics*, 320:1099–1134, 2017.

[LW09] Ka-Sing Lau and Xiang-Yang Wang. Self-similar sets as hyperbolic boundaries. *Indiana University Mathematics Journal*, 58(4):1777–1795, 2009.

Let τ be the first hitting time of any node in G then the entries of $\tilde{P}$ are given by

$$\tilde{p}_{x,y} = \mathbb{P}(X_\tau = y | X_0 = x),\ x \in G, y \in F.$$

Since we also have with $y_1, \dots, y_j \in H$

$$Q = \operatorname{diag}(\deg(y_1)^{-1}, \dots, \deg(y_j)^{-1}) \cdot A_{HG}$$

$$(R - I_{V_2}) = \operatorname{diag}(\deg(y_1)^{-1}, \dots, \deg(y_j)^{-1}) \cdot A_{HH}$$

it follows

$$B \overset{(2)}{=} A_{GG} - A_{GH}(R - I_{|H|})^{-1}Q.$$

The entries of B can then be calculated and are given by

$$b_{x,y} = \begin{cases} \sum_{z\sim x} \mathbb{P}(X_\tau = y | X_0 = z) & x \neq y, \\ -\deg(x) + \sum_{z\sim x} \mathbb{P}(X_\tau = x | X_0 = z) & x = y, \end{cases}$$

(see Figure 5) where $x, y \in G$, and

$$\mathcal{E}_G(u,u) = \operatorname{Tr}(\mathcal{E}_F|G)(u,u) = -u^T B u.$$

Induced Dirichlet Forms

Following [GY17] we construct a regular Dirichlet form on K. We start by defining a regular Dirichlet form on $L^2(X; m)$ where $m(x) = \left(\frac{c}{(4n+1)\lambda}\right)^{|x|}, c \in (0, \lambda)$ with $|x| = d(o, x)$.

$$\begin{cases} \mathcal{E}_X(u,u) = \frac{1}{2}\sum_{x,y\in X} a_{x,y}(u(x)-u(y))^2, \\ \mathcal{F}_X = \text{the } (\mathcal{E}_X)_1\text{-closure of } C_0(X). \end{cases}$$

We then are able to construct an active reflected Dirichlet space $\mathcal{F}_a^{ref}$ which yields

$$\begin{cases} \mathcal{E}^{ref}(u,u) = \frac{1}{2}\sum_{x,y\in X} a_{x,y}(u(x)-u(y))^2, \\ \mathcal{F}_a^{ref} = \left\{u \in L^2(X;m) : \mathcal{E}^{ref}(u,u) < \infty\right\}. \end{cases}$$

It can be shown that $(\mathcal{E}^{ref}, \mathcal{F}_a^{ref})$ is a Dirichlet form on $L^2(X; m)$ which is not necessarily regular. We therefore construct a regular representation $(\mathcal{E}_{\overline{X}}, \mathcal{F}_{\overline{X}})$ of this Dirichlet form on $L^2(\overline{X}; m)$, where $\overline{X}$ is the Martin compactification of X,

$$\begin{cases} \mathcal{E}_{\overline{X}}(u,u) = \frac{1}{2}\sum_{x,y\in X} a_{x,y}(u(x)-u(y))^2, \\ \mathcal{F}_{\overline{X}} = \left\{u \in C(\overline{X}) : \mathcal{E}_{\overline{X}}(u,u) < \infty\right\}. \end{cases}$$

This Dirichlet form is regular for $\lambda \in (rm^{-1}, m^{-1})$. We now are able to take the trace form to K to get a Dirichlet form $(\mathcal{E}_K, \mathcal{F}_K)$ on $L^2(K; \nu)$ where ν is the hitting distribution of $\partial_H X$. This Dirichlet form is then given by

$$\begin{cases} \mathcal{E}_K(u,u) = \int_K\int_K \frac{(u(x)-u(y))^2}{|x-y|^{\alpha+\beta}}\nu(dx)\nu(dy), \\ \mathcal{F}_K = \left\{u \in L^2(K;\nu) : \mathcal{E}_K(u,u) < \infty\right\}, \end{cases}$$

and regular for $\beta \in (\alpha, \beta^*)$, $\alpha = \dim_H(K)$. We finally show that for $\lambda \in (0, \lambda^*)$, or $\beta \in (\beta^*, \infty)$, $\mathcal{F}_K$ consists of constant functions. This yields

$$\dim_W(K) = \beta^* = \frac{\log((\lambda^*)^{-1})}{\log(r^{-1})}. \qquad (3)$$

Walk dimension of Vicsek sets

Let A_k be the conductance matrix of a square in V_k and A_{k+1} the conductance matrix of the part of V_{k+1} descending from the square. Then the energy $\mathcal{E}_k$ of this square in V_k gets contributions from the vertical edges connecting it to V_{k+1}

$$\mathcal{E}_v = \frac{5}{2C_1}r^2\mathcal{E}_k.$$

and from the $4n+1$ squares in the V_{k+1} who descend from it

$$\mathcal{E}_{k+1} = \frac{r}{(4n+1)\lambda}\mathcal{E}_k$$

since

$$A_{k+1}|_{V_k} = \frac{C_1}{((4n+1)\lambda)^{k+1}} \cdot \begin{pmatrix} -3r & r & r & r \\ r & -3r & r & r \\ r & r & -3r & r \\ r & r & r & -3r \end{pmatrix} = \frac{r}{(4n+1)\lambda}A_k.$$

Hence the triviality of $\mathcal{F}_K$ follows for $\lambda \in (0, rm^{-1})$ or $\beta \in (\beta^*, \infty)$, where

$$\beta^* = \frac{\log((4n+1)^{-1}r)}{\log(r)}.$$

Equation (3) yields

$$\dim_W(K) = \frac{\log(m^{-1}r)}{\log(r)} = 1 + \frac{\log(m)}{-\log(r)} = 1 + \dim_H(K).$$

Looking at the limit of the walk dimensions as n tends to infinity yields

$$\lim_{n\to\infty} \dim_W(K_n) = \lim_{n\to\infty} 1 + \frac{\log(4n+1)}{\log(2n+1)} = 2.$$