Concluding remarks

We consider Laplace operators on periodic discrete graphs perturbed by guides, i.e., graphs which are periodic in some directions and finite in other ones. We show that the spectrum of the Laplacian on the perturbed graph consists of the spectrum of the Laplacian on the unperturbed periodic graph and the additional so-called guided spectrum which is a union of a finite number of bands. We estimate the positions of the guided bands and their lengths in terms of geometric parameters of the graph. We also determine the asymptotics of the guided bands for guides with large multiplicity of edges.

Discrete Laplacians on graphs

Let $\Gamma = (V, \mathcal{E})$ be a connected infinite graph embedded into $\mathbb{R}^2$, $V$ is the set of its vertices and $\mathcal{E}$ is the set of its edges. Let $\mathcal{E}(V)$ be the Hilbert space of all square summable functions $f : V \to \mathbb{C}$, equipped with the norm $\|f\|_{2}(V) = \sum_{v \in V} |f(v)|^2 < \infty$.

We define the Laplacian $\Delta$ on $f \in \mathcal{E}(V)$:

$$ (\Delta f)(v) = \sum_{(v, \xi) \in \mathcal{E}} (f(v) - f(u)), \quad v \in V. $$

It is well known that $\Delta$ is self-adjoint and its spectrum satisfies:

$$ \sigma(\Delta) \subseteq [0, 2\kappa_v], \quad \kappa_v = \sup_{v \in V} \kappa_v < \infty, \quad \kappa_v \text{ is the degree of the vertex } v, \text{ i.e., the number of edges incident to } v. $$

Unperturbed periodic graphs

Let $\Gamma_0 \subset \mathbb{R}^2$ be a periodic graph, i.e., a graph satisfying the following conditions:

- There exists a basis $a_1, a_2$ in $\mathbb{R}^2$ such that $\Gamma_0$ is invariant under translations through the vectors $a_1, a_2$: $\Gamma_0 + a_1 = \Gamma_0, \quad s = 1, 2$;
- The fundamental graph $\Gamma_0 = \Gamma_0/\mathbb{Z}^2$ is finite.

We consider the Laplacian on the periodic graph $\Gamma_0$ as an unperturbed operator and denote it by $\Delta_0$. It is well known that the spectrum $\sigma(\Delta_0)$ of the Laplacian $\Delta_0$ on periodic graphs is a union of $\nu$ spectral bands $\sigma(\Delta_0)$:

$$ \sigma(\Delta_0) = \bigcup_{n=1}^{\nu} \sigma(\Delta_0) = \sigma_{\text{a.c.}}(\Delta_0) \cup \sigma_{\text{bnd}}(\Delta_0), $$

where $\nu = \# \mathcal{E}(V)$ is the number of vertices of the fundamental graph $\Gamma_0 = (V, \mathcal{E})$. The absolutely continuous spectrum $\sigma_{\text{a.c.}}(\Delta_0)$ consists of non-degenerate bands $\sigma_{\text{a.c.}}(\Delta_0)$, and $\sigma_{\text{bnd}}(\Delta_0)$ is the set of all flat bands (eigenvalues of infinite multiplicity).

$$ \sigma(\Delta_0) \subseteq [\kappa, \infty), \quad \kappa = \inf \sigma(\Delta_0) = 0, \quad \theta = \sup \sigma(\Delta_0) $$

Concluding remarks

- Roughly speaking, the guided spectrum may be any set above the unperturbed spectrum. Its total length may be arbitrarily large or arbitrarily small.
- The proof of all results is based on the decomposition of the Laplacian on the periodic graphs with guides into a direct integral and a precise representation of a fiber operator.

Periodic graphs with guides

Let $\Gamma_0 \subset \mathbb{R}^2$ be a periodic graph, and $S = \{0, 1\} \times \mathbb{R}$ be a fundamental strip (with respect to $a_1, a_2$).

$$ \Gamma_1 = (V_1, \mathcal{E}_1), \quad \Gamma_1 = (V_1, \mathcal{E}_1) \text{ is a finite connected decoration of } \Gamma_0 \cap S. $$

The spectrum of the Laplacian $\Delta$ on the unperturbed graph $\Gamma = \Gamma_0 \cup \Gamma_1$ has the form

$$ \sigma(\Delta) = \sigma(\Delta_0) \cup \sigma(\Delta_1), $$

where $\sigma(\Delta_0)$ is the spectrum of the Laplacian $\Delta_0$ on the unperturbed periodic graph $\Gamma_0$. The additional guided spectrum $\sigma(\Delta_1)$ may partly lie above the spectrum of the unperturbed Laplacian $\Delta_0$, on the spectrum of $\Delta_0$ and in the gaps of $\Delta_0$.

We consider the guided spectrum above the spectrum of $\Delta_0$:

$$ \sigma_{\text{g}}(\Delta_1) = \sigma_{\text{g}}(\Delta) = \sigma(\Delta_1) \cap \{g, +\infty\}, \quad \theta = \sup \sigma(\Delta_0). $$

Estimates of guided bands

The Laplacian $\Delta_1$ on the finite graph $\Gamma_1$ has $p$ positive eigenvalues: $\xi_0 \leq \ldots \leq \xi_p$.

Theorem 1 Each guided band $\sigma_{\text{g}}(\Delta)$ and their number $N$ satisfy

$$ \sigma_{\text{g}}(\Delta) \subset [\xi_j, \xi_j + \theta_0] \quad \left| \sigma_{\text{g}}(\Delta) \right| \leq 2\beta_j, $$

where $\beta_j = \max \beta_1, \ldots, \beta_N$, $N = \{1, \ldots, p\}$, $\beta_j = \max \{j \in \mathbb{N}_0 : \xi_j \leq \theta_0, \xi_j \leq \theta_0\}$, $\beta_j = \max \{j \in \mathbb{N}_0 : \xi_j \leq \theta_0, \xi_j \leq \theta_0\}$, $\beta_j = \max \{j \in \mathbb{N}_0 : \xi_j \leq \theta_0, \xi_j \leq \theta_0\}$, $\beta_j = \max \{j \in \mathbb{N}_0 : \xi_j \leq \theta_0, \xi_j \leq \theta_0\}$, $\beta_j = \max \{j \in \mathbb{N}_0 : \xi_j \leq \theta_0, \xi_j \leq \theta_0\}$.

Remarks. 1) The positions of the guided bands $\sigma_{\text{g}}(\Delta)$ on the Laplacian $\Delta_1$ on the perturbed graph $\Gamma = \Gamma_0 \cup \Gamma_1$ are defined by the eigenvalues of the Laplacian $\Delta_1$ on the finite graph $\Gamma_1$. The lengths of the guided bands are defined by the number of bands on the unperturbed periodic graph $\Gamma_0$.

2) For most graphs the number $\beta_p = 1$, then the guided band length $|\sigma(\Delta)\/\\sigma_{\text{g}}(\Delta)| \leq 2$ for all $\alpha = 1, \ldots, N$.

3) If the eigenvalues of $\Delta_1$ satisfy $\xi_j - \xi_{j+1} > \theta_0$ for all $j \in \mathbb{N}_0$, then the guided spectrum of the perturbed Laplacian $\Delta_1$ consists of exactly $p$ guided bands separated by gaps.

4) For any $\alpha > 0$ there exists a perturbed graph $\Gamma$ such that the length of each non-degenerate guided band $|\sigma(\Delta)| \leq \beta_p = \beta_p \leq \beta_p$, $\beta_p = \max \beta_p$, $\beta_p = \max \beta_p$, $\beta_p = \max \beta_p$, $\beta_p = \max \beta_p$.

Asymptotics of the guided bands

Let $\Gamma = (V_1, \mathcal{E}_1)$ be a finite graph obtained from $\Gamma_1 = (V_1, \mathcal{E}_1)$ considering each edge of $\Gamma_1$ to have the multiplicity $t \in \mathbb{N}$, and let $\Gamma_0 = (V_0, \mathcal{E}_0)$ be any periodic graph.

Theorem 2 Let all positive eigenvalues $\xi_j, \ldots, \xi_p$ of the Laplacian $\Delta_1$ on $\Gamma_1$ be distinct, and let $t \in \mathbb{N}$ be large enough. Then the guided spectrum of the Laplacian $\Delta$ on the perturbed graph $\Gamma = \Gamma_0 \cup \Gamma_1$ consists of $p = \#V_1 - 1$ guided bands $\sigma_{\text{g}}(\Delta) = [\xi_j(t), \xi_j(t)], j \in \mathbb{N}_0$, separated by gaps and

$$ \lambda_j(t) = t^2 \xi_j + O(1/t). $$

Remark. If $\beta_0 = 0$, then the obtained asymptotics take the form

$$ \lambda_j(t) = t^2 \xi_j + O(1/t), \quad \lambda_j(t) = O(1/t), \quad j \in \mathbb{N}_0, \quad \lambda_j(t) = O(1/t), \quad j \in \mathbb{N}_0, \quad \lambda_j(t) = O(1/t), \quad j \in \mathbb{N}_0. $$

For more details see