Spectrum of Laplacians on periodic graphs with guides

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Abstract

We consider Laplace operators on periodic discrete graphs perturbed by guides, i.e., graphs which are periodic in some directions and finite in other ones. We show that the spectrum of the Laplacian on the perturbed graph consists of the spectrum of the Laplacian on the unperturbed periodic graph and the additional so-called guided spectrum which is a union of a finite number of bands. We estimate the positions of the guided bands and their lengths in terms of geometric parameters of the graph. We also determine the asymptotics of the guided bands for guides with large multiplicity of edges.

Discrete Laplacians on graphs

Let $\Gamma = (V, \mathcal{E})$ be a connected infinite graph embedded into $\mathbb{R}^2$, V is the set of its vertices and $\mathcal{E}$ is the set of its edges.

Let $\ell^2(V)$ be the Hilbert space of all square summable functions $f : V \to \mathbb{C}$, equipped with the norm

$$\|f\|^2_{\ell^2(V)} = \sum_{v\in V} |f(v)|^2 < \infty.$$

We define the Laplacian Δ on $f \in \ell^2(V)$:

$$(\Delta f)(v) = \sum_{(v,u)\in\mathcal{E}} \big(f(v) - f(u)\big), \qquad v \in V.$$

It is well known that Δ is self-adjoint and its spectrum satisfies:

$$\sigma(\Delta) \subset [0, 2\varkappa_+], \quad \text{where} \quad \varkappa_+ = \sup_{v\in V} \varkappa_v < \infty,$$

$\varkappa_v$ is the *degree* of the vertex v, i.e., the number of edges incident to v.

Unperturbed periodic graphs

Let $\Gamma_0 \subset \mathbb{R}^2$ be a *periodic graph*, i.e., a graph satisfying the following conditions:

- there exists a basis a_1, a_2 in $\mathbb{R}^2$ such that Γ_0 is invariant under translations through the vectors a_1, a_2: $\Gamma_0 + a_s = \Gamma_0$, $s = 1, 2$.
- the fundamental graph $\Gamma_* = \Gamma_0/\mathbb{Z}^2$ is finite.

Γ_* is a graph on the 2-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$.

We consider the Laplacian on the periodic graph Γ_0 as an *unperturbed operator* and denote it by Δ_0.

It is well known that the spectrum $\sigma(\Delta_0)$ of the Laplacian Δ_0 on periodic graphs is a union of ν spectral bands $\sigma_n(\Delta_0)$:

$$\sigma(\Delta_0) = \bigcup_{n=1}^{\nu} \sigma_n(\Delta_0) = \sigma_{ac}(\Delta_0) \cup \sigma_{fb}(\Delta_0),$$

where $\nu = \#V_*$ is the number of vertices of the fundamental graph $\Gamma_* = (V_*, \mathcal{E}_*)$.

The absolutely continuous spectrum $\sigma_{ac}(\Delta_0)$ consists of non-degenerate bands $\sigma_n(\Delta_0)$; $\sigma_{fb}(\Delta_0)$ is the set of all flat bands (eigenvalues of infinite multiplicity).

$$\sigma(\Delta_0) \subset [0, \varrho], \quad \inf \sigma(\Delta_0) = 0, \quad \varrho = \sup \sigma(\Delta_0)$$

Concluding remarks

- Roughly speaking, the guided spectrum may be any set above the unperturbed spectrum. Its total length may be arbitrarily large or arbitrarily small.
- The proof of all results is based on the decomposition of the Laplacian on the periodic graphs with guides into a direct integral and a precise representation of a fiber operator.

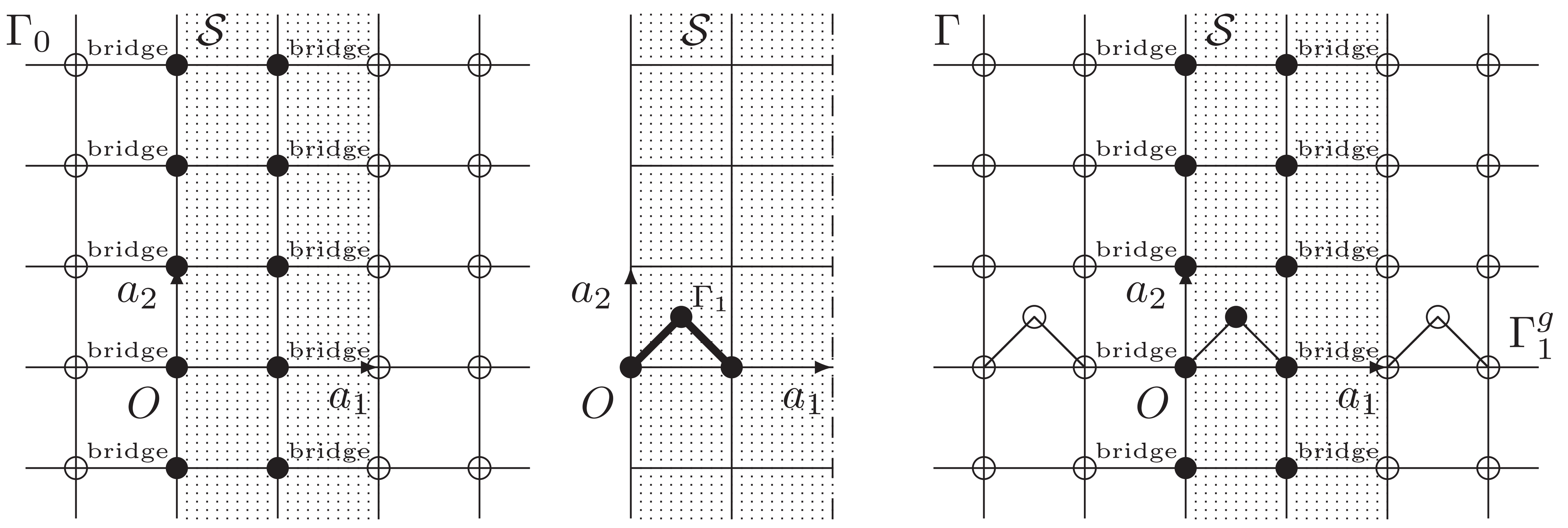
Periodic graphs with guides

$\mathcal{S} = [0, 1) \times \mathbb{R}$ is *a fundamental strip* (with respect to a_1, a_2).

$\Gamma_1 = (V_1, \mathcal{E}_1)$ is a finite connected decoration of $\Gamma_0 \cap \mathcal{S}$.

$\Gamma_1^g = \bigcup_{m\in\mathbb{Z}} (\Gamma_1 + ma_1)$ is *a guide* induced by Γ_1.

$\Gamma = \Gamma_0 \cup \Gamma_1^g$ is *a periodic graph with a guide* Γ_1^g or *a perturbed graph*.



Bridges are edges of Γ_0 connecting the vertices from $\mathcal{S}$ (black points) with the vertices outside $\mathcal{S}$ (white points).

Spectrum of the Laplacian on the perturbed graph

The spectrum of the Laplacian Δ on the perturbed graph $\Gamma = \Gamma_0 \cup \Gamma_1^g$ has the form

$$\sigma(\Delta) = \sigma(\Delta_0) \cup \sigma^g(\Delta),$$

where $\sigma(\Delta_0)$ is the spectrum of the Laplacian Δ_0 on the unperturbed periodic graph Γ_0.

The additional *guided* spectrum $\sigma^g(\Delta)$ may partly lie above the spectrum of the unperturbed Laplacian Δ_0, on the spectrum of Δ_0 and in the gaps of Δ_0.

We consider the guided spectrum **above the spectrum of** Δ_0:

$$\sigma^g_+(\Delta) = \sigma^g(\Delta) \cap [\varrho, +\infty), \quad \varrho = \sup \sigma(\Delta_0),$$

$$\sigma^g_+(\Delta) = \bigcup_{j=1}^{N} \sigma^g_j(\Delta) = \sigma^g_{ac}(\Delta) \cup \sigma^g_{fb}(\Delta),$$

$N \le p = \#V_1 - 1$,

$\sigma^g_{ac}(\Delta)$ is the absolutely continuous part (a union of non-degenerate bands $\sigma^g_j(\Delta)$),

$\sigma^g_{fb}(\Delta)$ is the set of all degenerate bands $\sigma^g_j(\Delta)$ (eigenvalues of infinite multiplicity).

Estimates of guided bands

The Laplacian Δ_1 on the finite graph Γ_1 has p positive eigenvalues: $\xi_p \le \ldots \le \xi_1$.

Theorem 1 *Each guided band $\sigma^g_j(\Delta)$ and their number N satisfy*

$$\sigma^g_j(\Delta) \subset [\xi_j, \xi_j + \varrho], \qquad |\sigma^g_j(\Delta)| < 2\beta_+,$$

$$N \ge \#\{j \in \mathbb{N}_p : \xi_j > \varrho\}, \qquad \mathbb{N}_p = \{1, \ldots, p\},$$

where $\beta_+ = \max_{v\in V_0} \beta_v$, β_v is the number of bridges of the unperturbed periodic graph $\Gamma_0 = (V_0, \mathcal{E}_0)$ at the vertex $v \in V_0$.

Remarks. 1) The positions of the guided bands $\sigma^g_j(\Delta)$ of the Laplacian Δ on the perturbed graph $\Gamma = \Gamma_0 \cup \Gamma_1^g$ are defined by the eigenvalues of the Laplacian Δ_1 on the finite graph Γ_1. The lengths of the guided bands are defined by the number of bridges on the unperturbed periodic graph Γ_0.

2) For most of graphs the number $\beta_+ = 1$, then the guided band length $|\sigma^g_j(\Delta)| \le 2$ for all $j = 1, \ldots, N$, but for specific graphs β_+ may be any given positive integer number.

3) If the eigenvalues of Δ_1 satisfy $\xi_j - \xi_{j+1} > \varrho$ for all $j \in \mathbb{N}_{p-1}$ and $\xi_p > \varrho$, then the guided spectrum of the perturbed Laplacian Δ consists of exactly p guided bands separated by gaps.

4) For any $\varepsilon > 0$ there exists a perturbed graph Γ such that the length of each non-degenerate guided band $|\sigma^g_j(\Delta)| > 2\beta_+ - \varepsilon$, $j = 1, \ldots, N$.

Asymptotics of the guided bands

Let $\Gamma_t = (V_1, \mathcal{E}_t)$ be a finite graph obtained from $\Gamma_1 = (V_1, \mathcal{E}_1)$ considering each edge of Γ_1 to have the multiplicity $t \in \mathbb{N}$, and let $\Gamma_0 = (V_0, \mathcal{E}_0)$ be any periodic graph.

Theorem 2 *Let all positive eigenvalues $\xi_p < \ldots < \xi_1$ of the Laplacian Δ_1 on Γ_1 be distinct, and let $t \in \mathbb{N}$ be large enough. Then the guided spectrum of the Laplacian Δ on the perturbed graph $\Gamma = \Gamma_0 \cup \Gamma_t^g$ consists of $p = \#V_1 - 1$ guided bands $\sigma^g_j(\Delta) = [\lambda^-_j(t), \lambda^+_j(t)]$, $j \in \mathbb{N}_p$, separated by gaps and*

$$\lambda^\pm_j(t) = t\xi_j + C^\pm_j + O(1/t),$$

$$|\sigma^g_j(\Delta)| = C_j + O(1/t), \quad \textit{as} \quad t \to \infty,$$

$$C_j = C^+_j - C^-_j \le 2\beta^0_+, \quad \beta^0_+ = \max_{v\in V_0\cap V^g_t} \beta^0_v.$$

Here $C^\pm_j$ are some constants, β^0_v is the number of bridges at v connecting vertices from $V_0 \cap V^g_t$, where V^g_t is the vertex set of the guide Γ^g_t.

Remark. If $\beta^0_+ = 0$, then the obtained asymptotics take the form

$$\lambda^\pm_j(t) = t\xi_j + O(1/t), \qquad |\sigma^g_j(\Delta)| = O(1/t),$$

$j \in \mathbb{N}_p$, and the total length of the guided spectrum of the Laplacian Δ satisfies $|\sigma^g_+(\Delta)| = O(1/t)$ as $t \to \infty$.

For more details see

Korotyaev E., Saburova N. *Laplacians on periodic graphs with guides*, J. Math. Anal. Appl. **455** (2017), no. 2, 1444–1469.