Objective

Multiresolution analysis arising from Coalescence Hidden-variable Fractal Interpolation Functions (CHFIFs) is developed. This is done via the following steps:

- Construction of a CHFIF
- Vector space of CHFIFs and determining dimension of the vector space
- Constructing Riesz bases for vector subspaces $\mathbb{V}_k, k \in \mathbb{Z}$ of above vector space
- Developing Multiresolution analysis in terms of nested sequences of vector subspaces $\mathbb{V}_k, k \in \mathbb{Z}$.

Introduction

The theory of multiresolution analysis provides a powerful method to construct wavelets having far reaching applications in analyzing signals and images. In [Hardin D.P. et al., 1992], multiresolution analysis of $L^2(\mathbb{R})$ were generated from certain classes of Affine Fractal Interpolation Functions (AFIFs). A few years later, Donovan et al [Donovan G.C. et al., 1996] constructed orthogonal compactly supported continuous wavelets using multiresolution analysis arising from AFIFs. The interrelations among AFIFs, Multiresolution Analysis and Wavelets are treated in [Massopust P., 2010]. It is desirable [Christopher Torrence and Gilbert P. Compo, 1998] that the wavelet function should reflect the features present in the original function but AFIF based wavelets generally cannot exhibit satisfactorily the features of functions simulating natural objects or outcome of scientific experiments that are partly self-affine and partly non-self-affine. The Coalescence Hidden Variable Fractal Interpolation Functions (CHFIFs) introduced in [Chand A.K.B. and Kapoor G.P., 2007] are ideally suited for such purposes. However, multiresolution analysis of $L_2(\mathbb{R})$ based on CHFIFs has hitherto remained unexplored. In the present work, such a multiresolution analysis using CHFIFs as basis functions is developed.

Construction of a CHFIF

- Given data $\{(x_n, y_n) \in \mathbb{R}^2 : n = 0, 1, ..., N\}$
- Generalized data $\{(x_n, y_n, z_n) \in \mathbb{R}^3 : n = 0, 1, \dots, N\}$
- $[x_0, x_N] = I, [x_{n-1}, x_n] = I_n, n = 1, 2, \dots, N$
- L_n contraction maps from I to $I_n = [x_{n-1}, x_n]$
- α_n, γ_n Free Parameters, β_n Constrained Parameters, p_n, q_n continuous functions

A CHFIF is constructed such that

1CHFIF:

 $f_1(x) = \alpha_n f_1(L_n^{-1}(x)) + \beta_n f_2(L_n^{-1}(x)) + p_n(L_n^{-1}(x)), \ x \in I_n, n = 1, \dots, N$ passing through $\{(x_n, y_n) : n = 0, 1, \dots, N\}$

 $2f_2(x) = \gamma_n f_2(L_n^{-1}(x)) + q_n(L_n^{-1}(x)), x \in I_n, n = 1, ..., N$ passes through $\{(x_n, z_n) : n = 0, 1, \dots, N\}$

Definition of Multiresolution Analysis

A Multiresolution Analysis consists of closed linear subspaces V_k of $L^2(\mathbb{R})$ that satisfy:

- $\ldots \supseteq V_{-1} \supseteq V_0 \supseteq V_1 \supseteq \ldots$
- $\bigcap_{k\in\mathbb{Z}}V_k=\{0\}$
- $\operatorname{clos}_{L^2} \bigcup_{k \in \mathbb{Z}} V_k = L^2(\mathbb{R})$
- $f \in V_0 \Leftrightarrow f(2^{-k} \cdot -l) \in V_k, \ k, l \in \mathbb{Z}.$
- There exists a function $\phi \in L^2(\mathbb{R})$ such that $\phi_{k,l}, l \in \mathbb{Z}$, defined by
- $\phi_{k,l}(x) = 2^{-k/2}\phi(2^{-k}x-l)$, form a Riesz basis of V_k for each $k \in \mathbb{Z}$.

A set $B = \{\phi_1, \phi_2, \dots, \phi_M\}, M \in \mathbb{N}, M > 1$, of scaling functions is said to generate a multiresolution analysis of $L^2(\mathbb{R})$ if ϕ and $\phi_{k,l}$ in the above definition of MRA are replaced by ϕ_i and $\phi_{i,k,l}$, $i = 1, 2, \ldots, M$ respectively.

Multiresolution analysis based on Coalescence Hidden-variable Fractal Interpolation Functions

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Vector space of CHFIFs and its dimension

Let the set \mathcal{S}_0 consisting of functions $f : I \to \mathbb{R}^2$ be defined as $\mathcal{S}_0 = \{f : f = (f_1, f_2), f_1 \text{ is a CHFIF passing through } \{(x_i, y_i) \in \mathbb{R}^2 :$ $i = 0, 1, \ldots, N$ and f_2 is an AFIF passing through $\{(x_i, z_i) \in \mathbb{R}^2 : i =$ $0, 1, \ldots, N$ }. Then, S_0 is a vector space, with usual point-wise addition and scalar multiplication. The dimension of \mathcal{S}_0 is 2(N+1).

Definition

Let \mathcal{S}_0^1 be the set of functions $f_1 : I \to \mathbb{R}$ that are first components of functions $f \in S_0$. The space of CHFIFs is the set S_0^1 together with the maximum metric $d^*(f,g) = \max_{x \in I} |f(x) - g(x)|$. The dimension of \mathbb{S}_0 is 2N.

Notations

 $y_0 = (1, r_1, \dots, r_{N-1}, 0), \quad y_N = (0, s_1, \dots, s_{N-1}, 1),$ $y_i = (0, \dots, 1, \dots, 0), \quad i = 1, \dots, N-1,$ $y_{N+1+i} = (0, u_{i,1}, \dots, u_{i,N-1}, 0), i = 0, \dots, N;$ $z_i = (0, \dots, 0), \quad z_{N+1+i} = (0, \dots, 1, \dots, 0), \quad i = 0, \dots, N$

• $f_i = (f_{i,1}, f_{i,2}) \ i = 0, \dots, 2N+1$

• $f_{i,1}$ passes through the points $\{(x_k, (y_i)_k), k = 0, \ldots, N\}$

• $f_{i,2}$ passes through the points $\{(x_k, (z_i)_k), k = 0, \ldots, N\}$

	Main Results	[Kapoor G
• $\mathbb{V}_0 = \operatorname{clos}_{L^2(\mathbb{R})} \operatorname{span} \{ \phi_{i,1}(\cdot - l) : i = 1 \}$	$,\ldots,2N-1,\ l\in\mathbb{Z}$	
• $\{\phi_{i,1}\}_{i=1}^{2N-1}$ generates a continuous, co	mpactly supported m	nultiresolution anal

Outline of Proof of Result 1

• $g_1 \in \mathbb{V}_0$ for some $g = (g_1, g_2) \in \mathbb{V}_0$

• g has unique expansion in terms of the functions $f_i = (f_{i,1}, f_{i,2}), i = 0, \ldots 2N + 1$, and their integer translates

• g_1 has a unique expansion in terms of the functions $\phi_{i,1}$, $i = 1, \ldots, 2N - 1$, and their integer translates • \mathbb{V}_0 and \mathbb{V}_0 is closed.

Outline of Proof of Result 2

(i)... $\supseteq \mathbb{V}_{-1} \supseteq \mathbb{V}_0 \supseteq \mathbb{V}_1 \supseteq \ldots$

• $g_1 \in \mathbb{V}_1$ for some $g = (g_1, g_2) \in \widetilde{\mathbb{V}}_1$

• $G = \bigcup_{i=1}^{N} w_i(G)$

•
$$w_j(G) = \bigcup_{i=1}^{N} w_j \circ w_i \circ w_j^{-1}(w_j(G))$$

- $w_j(G)$ is graph of $g|_{[j-1,j)}$
- $g \in \mathbb{V}_0, g_1 \in \mathbb{V}_0$

(ii)
$$\underset{k \in \mathbb{Z}}{\cap} \mathbb{V}_k = \{0\}$$

• $g_{\chi_{J_0}}(x) = \begin{cases} g(x) \ x \in J_0 = [0, 1] \\ 0 \ x \notin J_0 \end{cases}, g \in \mathbb{V}_0$ • $\|g_{\chi_{J_n}}\|_{\infty} \leq c \|g_{\chi_{J_n}}\|_{L^2(\mathbb{R})}, \ J_n = [n, n+1], n = 0, 1, \dots$ $\|g\|_{\infty} \leq \sup_{n} \|g_{\chi_{J_n}}\|_{\infty} \leq c \sum_{n \in \mathbb{Z}} \|g_{\chi_{J_n}}\|_{L^2(\mathbb{R})} = c \|g\|_{L^2(\mathbb{R})}$ $\|g\|_{\infty} \leq cN^{k/2} \|g\|_{L^2(\mathbb{R})} \text{ for all } g \in \mathbb{V}_k$ $\square \bigcap_{k \in \mathbb{Z}} V_k = \{0\}$

Definitions

 $\mathbb{V}_0 = \{f : f = (f_1, f_2), f_1|_{[m-1,m)}$ is a CHFIF $f_2|_{[m-1,m)}$ is a FIF, $m \in \mathbb{Z}$, $f_1, f_2 \in L^2(\mathbb{R}) \cap C_0(\mathbb{R})$ and $\tilde{\mathbb{V}}_k = \{f : f(N^{-k} \cdot) \in \tilde{\mathbb{V}}_0\}.$ Example

 $f = (f_1, f_2)$ with $x_0 = 0, x_N = 1, f(x_0) = (0, 0) = f(x_N)$ and set f(x) = (0,0) for $x \notin I$.

Definition

 $\mathbb{V}_0 = \{f_1 : f_1 \text{ is the first component of some } f = (f_1, f_2) \in \mathbb{V}_0\}$ and $\mathbb{V}_k =$ $\{f_1: f_1(N^{-k}\cdot) \in \mathbb{V}_0\}.$

Constructions

- $< f_{i,1}, f_{0,1} >= 0$ and $< f_{i,1}, f_{N,1} >= 0$
- $< f_{N+1+j,1}, f_{i,1} >= 0, < f_{N+1+i,1}, f_{0,1} >= 0 \text{ and } < f_{N+1+i,1}, f_{N,1} >= 0$ • Gram-Schmidt Process : $\{\phi_{i,1}\}_{i=1}^{2N-1} \subset \mathbb{V}_0, i \neq N$
- Set

$$\phi_{N,1} = \begin{cases} f_{N,1}(x) & x \in [0,1) \\ f_{0,1}(x-1) & x \in [1,2) \\ 0 & \text{otherwise} \end{cases}$$

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lysis of $L_2(\mathbb{R})$

$$(iii) \operatorname{clos}_{L^{2}(\mathbb{R})} \bigcup_{k \in \mathbb{Z}} \mathbb{V}_{k} = L^{2}(\mathbb{R})$$

$$1 = \sum_{k} \left(\sum_{i=1}^{N-1} C_{k,i} f_{i,1}(x-k) + \phi_{N}(x-k) + \sum_{i=1}^{N-1} D_{k,i} f_{N+1+i,1}(x-k) \right)$$

$$C_{k,i} = \left(1 - r_{i} - s_{i} - \sum_{j=1}^{N-1} u_{j,i} z_{j} \right) \text{ and } D_{k,i} = z_{i}, \quad i = 1, \dots, N-1.$$

$$(iv) \phi_{i,1}, \ i = 1, \dots, 2N-1 \text{ form a Riesz basis of } \mathbb{V}_{0}.$$

$$\frac{1}{\|\phi_{N,1}\|^{2}} \left((I_{I} | f_{0,1}(x)|^{2} dx)^{1/2} - (I_{I} | f_{0,1}(x)| | f_{N,1}(x) | dx)^{1/2} \right).$$

$$A = \sqrt{\tau} \|\phi_{N,1}\|_{L^{2}(\mathbb{R})} \text{ and } B = 3 \|\phi_{N,1}\|_{L^{2}(\mathbb{R})}$$

$$A \|c\|_{l^{2}} \leq \|\sum c_{i}\phi_{N,1} (\cdot - i)\|_{L^{2}(\mathbb{R})} \leq B \|c\|_{l^{2}}$$

• $\phi_{i,1}, i = 1, \ldots, 2N - 1, i \neq N$, and their integer translates are mutually orthogonal

Remarks

It follows that the set $\{\hat{\phi}_{i,1} : i = 1, 2, \dots, 2N - 1\} \subset \mathbb{V}_0$, where $\hat{\phi}_{i,1}(x) = \phi_{i,1}(x) / \|\phi_{i,1}\|_{L^2}, \ i = 1, 2, \dots, 2N-1, \text{ actually generates a contin-}$ uous, compactly supported multiresolution analysis of $L_2(\mathbb{R})$ by orthonormal functions.

Conclusion

In this paper, multiresolution analysis arising from Coalescence Hiddenvariable Fractal Interpolation Functions is developed, since CHFIF based wavelets would generally more satisfactorily preserve the features of the functions simulating natural objects or outcome of scientific experiments that are partly self-affine and partly non-self-affine than AFIF based wavelets. The availability of a larger set of free variables and constrained variables with CHFIF in multiresolution analysis based on CHFIFs provides more control in reconstruction of functions in $L_2(\mathbb{R})$ than that provided by multiresolution analysis based only on affine FIFs.

Additional Information

- Orthogonal bases consisting of dilations and translations of scaling functions, for the vector subspaces $\mathbb{V}_k, k \in \mathbb{Z}$, consisting of certain CHFIFs in $L_2(\mathbb{R}) \cap C_b(\mathbb{R})$, have been constructed.
- As a natural follow-up, the orthogonality of these scaling functions have been used to construct compactly supported continuous orthonormal wavelets.

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Acknowledgements

The second author thanks CSIR for Research Grant No: 9/92(417)/2005-EMR-I for the present work.

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