On a norm resolvent convergence result for resistance forms on the Diamond Lattice

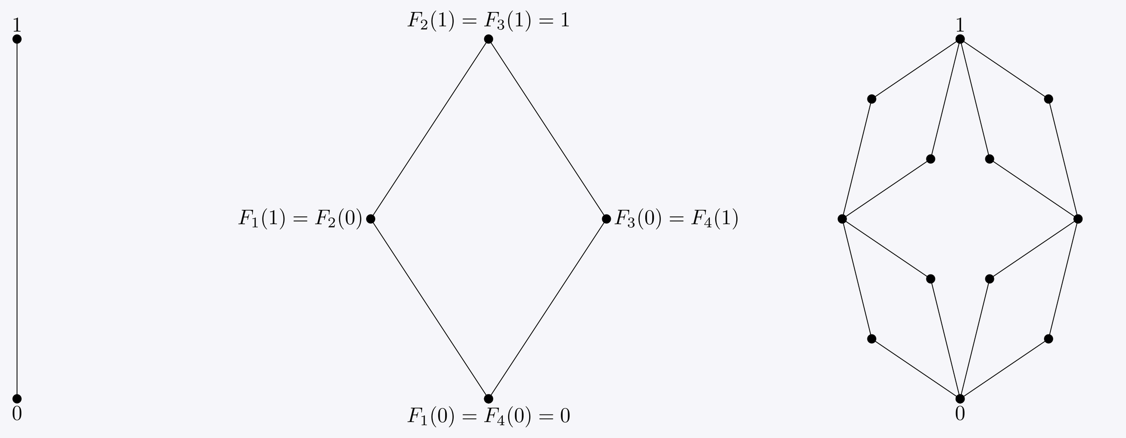
Olaf Post — olaf.post@uni-trier.de

Jan Simmer — simmer@uni-trier.de

FGS 6 — Universität Trier

Definitions and notation

Diamond Lattice Fractal. Let (X,d) be a compact metric space containing $0 \neq 1$. The *Diamond Lattice Fractal* D is the unique self-similar set w.r.t four contractive similarities $F_j \colon X \to X$ with contraction ratio $1/2$ s.th.:



For $w = w_1 \dots w_m \in W_m := \{1,2,3,4\}^m$ we define $w \mapsto F_w(D) := F_{w_1} \circ \dots \circ F_{w_m}(D)$.

The approximating graphs. We define a sequence $G_m = (V_m, E_m)$, where G_0 is the graph with vertices V_0 and only one edge $\{0,1\}$ and recursively $V_m := \bigcup_{w \in W_m} F_w(V_0)$ with edges $E_m := \{\{x,y\} : x \sim_m y\}$, where $x \sim_m y$ iff $x \neq y \in V_m$ and there exists a word $w \in W_m$ such that $x, y \in F_w(D)$.
Moreover, there exists a compatible sequence of energy forms $\{\mathcal{E}_m\}_{m \in \mathbb{N}_0}$ given by

$$\mathcal{E}_m(f) := \frac{1}{2} \sum_{x \sim_m y} |f(x) - f(y)|^2$$

for $f, g \in \ell(V_m) := \{f \colon V_m \to \mathbb{C}\}$. Hence $\mathcal{E}(u) := \lim_{m \to \infty} \mathcal{E}_m(u|_{V_m})$ exists in $[0,\infty]$ for $u \colon V_\star := \bigcup_{m \geq 0} V_m \to \mathbb{C}$ and $(\mathcal{E}, \operatorname{dom} \mathcal{E})$ is a *resistance form* where

$$\operatorname{dom} \mathcal{E} := \Big\{ u \colon V_\star \to \mathbb{C} : \sup_m \mathcal{E}_m(u|_{V_m}) < \infty \Big\}$$

Let μ be the homogeneous self-similar Hausdorff measure on D with weights $\mu_i = 1/4$. Then $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ induces a self-similar local regular Dirichlet form in $\mathrm{L}_2(D,\mu)$ (cf. [1, Thm. 4.3]).

Defining magnetic potentials. Let $\mathcal{H}$ be the Hilbert module of 1-forms associated with $(\mathcal{E}, \operatorname{dom}\mathcal{E})$. Then there is a derivation $\partial \colon \operatorname{dom}\mathcal{E} \to \mathcal{H}$ such that $\mathcal{E}(u) = \|\partial u\|^2_{\mathcal{H}}$ (cf. [2]). We denote the finite-dimensional subspace generated by m-harmonic functions by

$$\mathcal{H}_m := \Big\{ \sum_{w \in W_m} \partial A_w \mathbb{1}_{D_w} : A_w \ m\text{-harmonic} \Big\}.$$

For each real valued $a \in \mathcal{H}$, we define a *magnetic energy form* $(\mathcal{E}^a, \operatorname{dom}\mathcal{E})$ in $\mathrm{L}_2(D,\mu)$ by

$$\mathcal{E}^a(u) := \big\|(\partial + \mathrm{i}a)u\big\|^2_{\mathcal{H}}.$$

If $a = \sum_{w \in W_m} A_w \otimes \mathbb{1}_w \in \mathcal{H}_m$, on G_m, we define

$$\mathcal{E}^a_m(f) = \sum_{w \in W_m} \sum_{x,y \in F_w(V_0)} \big|f_{a,w}(x) - f_{a,w}(y)\big|^2,$$

where $f_{a,w}(x) := f(x)\mathrm{e}^{\mathrm{i}A_w(x)}$.

References

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[3] O. Post, Lecture Notes in Mathematics, vol. 2039, Springer, Heidelberg, 2012.

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Generalised norm resolvent convergence

Let $\mathscr{H}_m$ and $\mathscr{H}_\infty$ be (distinct) separable Hilbert spaces with energy forms $(\mathcal{E}_m, \mathscr{H}^1_m)$ resp. $(\mathcal{E}_\infty, \mathscr{H}^1_\infty)$ (i.e. non-negative, closed quadratic forms) and with associated operators Δ_m resp. Δ_∞. We denote $\|.\|^2_{\mathscr{H}^1_m} := \|.\|^2_1 := \|.\|^2_{\mathscr{H}_m} + \mathcal{E}_m(.)$.

Definition 1 *Let $\delta_m > 0$. Then, $\mathcal{E}_m$ and $\mathcal{E}_\infty$ are δ_m-quasi-unitarily equivalent (δ_m-que) if there exist operators $J_m \colon \mathscr{H}_m \to \mathscr{H}_\infty$, bounded by 1 with $J(\mathscr{H}^1_m) \subset \mathscr{H}^1_\infty$ and $J'^1 \colon \mathscr{H}^1_\infty \to \mathscr{H}^1_m$ s.th.*

$$\|f - J^\star_m J_m f\|_{\mathscr{H}_m} \leq \delta_m \|f\|_{\mathscr{H}^1_m} \qquad (1)$$

$$\|u - J_m J^\star_m u\|_{\mathscr{H}_\infty} \leq \delta_m \|u\|_{\mathscr{H}^1_\infty} \qquad (2)$$

$$\|J'^1_m u - J^\star_m u\|_{\mathscr{H}_m} \leq \delta_m \|u\|_{\mathscr{H}^1_\infty} \qquad (3)$$

$$\big|\mathcal{E}_\infty(J_m f, u) - \mathcal{E}_m(f, J'^1_m u)\big| \leq \delta_m \|f\|_1 \|u\|_1 \qquad (4)$$

The above definition gives us some flexibility which is useful in our application:

Lemma 1 *If (1),(2),(4) hold with $\delta'_m > 0$ and*

$$\exists \delta''_m > 0 : \quad \|u - J_m J'^1_m u\|_{\mathscr{H}_\infty} \leq \delta''_m \|u\|_{\mathscr{H}^1_\infty},$$

then $\mathcal{E}_m$ and $\mathcal{E}_\infty$ are δ_m-que where

$$\delta_m := \delta'_m + (1 + \delta_m)\delta_m.$$

Moreover, the notion is *transitive* in the following sense: If $\mathcal{E}_m$ and $\mathcal{E}_\infty$ are δ_m-que and $\mathcal{E}_\infty$ and $\widetilde{\mathcal{E}}$ are δ_∞-que then $\mathcal{E}_m$ and $\widetilde{\mathcal{E}}$ are $\widetilde{\delta}$-que for some $\widetilde{\delta}$ that can be determined explicitly.

Proposition 1 *Let $\eta \colon [0,\infty) \to \mathbb{C}$ be holomorphic in a neighbourhood U of $\sigma(\Delta_\infty)$ and s.th. $\lim_{\lambda \to \infty} (\lambda + 1)^{1/2} \eta(\lambda)$ exists. Then there exists a constant $C := C_{\eta,U} > 0$ s.th.*

$$\|\eta(\Delta_\infty) - J_m \eta(\Delta_m) J^\star_m\| \leq C\delta_m$$

$$\|\eta(\Delta_m) - J^\star_m \eta(\Delta_\infty) J_m\| \leq C\delta_m.$$

For example if $\eta_t(\lambda) = \mathrm{e}^{-t\lambda}$ then the above proposition is about the norm convergence of the heat operators. If $\eta = \mathbb{1}_I$ for some interval I s.th. $\partial I \cap \sigma(\Delta_\infty) = \emptyset$ then we conclude the norm convergence of the spectral projections.

Proposition 2 *Let $\lambda_k(\Delta_m)$ resp. $\lambda_k(\Delta_\infty)$ be the k-th eigenvalue of Δ_m resp. Δ_∞. Then, for all $m \in \mathbb{N}$,*

$$|\lambda_k(\Delta_m) - \lambda_k(\Delta_\infty)| \leq C_k \delta,$$

s.th. $\dim \mathscr{H}_m \geq k$; C_k only depends on $\lambda_k(\Delta_\infty)$.

Moreover, one can show that eigenfunctions converge in energy norm, i.e., if Φ_∞ is an eigenfunction of Δ_∞ isolated eigenvalue $\lambda(\Delta_\infty)$ then there exist a constant $C > 0$ depending only on λ_∞ and the radius of the disk and an eigenfunction Φ_m of Δ_m such that $\|J_m \Phi_m - \Phi_\infty\|_{\mathscr{H}^1_\infty} \leq C\delta$.
See [3, 4] for more details on the topic and proofs.

Our main result

Let μ be the self-similar Hausdorff measure on D and $\mathscr{H}_\infty := \mathrm{L}_2(D,\mu)$. We define the approximating measure $\mu_m = \{\mu_m(x)\}_{x \in V_m}$ on G_m by

$$\mu_m(x) := \int_D \psi_{x,m}(t) \,\mathrm{d}\mu(t),$$

where $\psi_{x,m} \colon D \to [0,1]$ is the m-harmonic function with boundary values $\mathbb{1}_{\{x\}}$ in V_m and we set $\mathscr{H}_m := \ell_2(V_m, \mu_m)$ with norm given by

$$\|f\|^2_{\mathscr{H}_m} := \sum_{x \in V_m} \mu_m(x) |f(x)|^2.$$

Theorem 1 *Let $a \in \mathcal{H}_m$ be real valued. Then $\mathcal{E}^a$ and $\mathcal{E}^a_m$ are δ_m-quasi-unitarily equivalent where*

$$\delta_m = (1 + \sqrt{2}) \cdot 2^{-m}.$$

Sketch of the proof:

- Define the operator $J_m \colon \mathscr{H}_m \to \mathscr{H}_\infty$ by

$$J_m f = \sum_{x \in V_m} f(x) \psi^a_{x,m},$$

$$\psi^a_{x,m} = \sum_{w \in W_{x,m}} \mathrm{e}^{\mathrm{i}A_w(x) - \mathrm{i}A_w} \psi_{x,m}|_{D_w \setminus V_m}$$

  where $\psi^a_{x,m}$ can be continuously extended to D. By the Cauchy-Young inequality we see that J_m is bounded by 1 and since $\psi_{x,m} \in \mathscr{H}^1_\infty := \operatorname{dom}\mathcal{E}_\infty$, we have $J_m(\mathscr{H}^1_m) \subset \mathscr{H}^1_\infty$. Then $J^\star_m \colon \mathscr{H}_\infty \to \mathscr{H}_m$,

$$J^\star_m u(y) = \frac{1}{\mu_m(y)} \langle u, \psi^a_{y,m} \rangle_{\mathscr{H}_\infty} \qquad (y \in V_m).$$

- Let $J'^1_m \colon \mathscr{H}^1_\infty \to \mathscr{H}^1_m$ be the evaluation, i.e., $J'^1_m u(y) = u(y)$, $y \in V_m$. This makes sense because functions in $\mathscr{H}^1_\infty$ are continuous.

- Next, we need to verify (1)–(4) from Definition 1. For (1) we first compute

$$f - J^\star_m J_m f(y) = \frac{1}{\mu_m(y)} \sum_{w \in W_m} \sum_{x \in F_w(V_0)} (f_{a,w}(x) - f_{a,w}(y)) \cdot \langle \psi_{x,m}|_{D_w}, \psi_{y,m}|_{D_w} \rangle_{\mathscr{H}_\infty}.$$

  Then, we can estimate in norm by applying the Cauchy-Schwarz inequality. In a similar way we treat the inequality (2), using the Hölder 1/2-estimate (w.r.t. the resistance metric associated with $\mathcal{E}$).

- Instead of proving (3) we apply Lemma 1. This helps us to omit an eigenvalue discussion which we would face in the unmodified version of the estimation. Note that the modification however changes the error term δ_m.

- Since $\psi^a_{x,m}$ is m-harmonic w.r.t. $\mathcal{E}^a$, the last inequality (4) is actually an equality, i.e.,

$$\mathcal{E}^a_m(f, J'^1_m u) = \mathcal{E}^a_\infty(J_m f, u).$$

By the above and the transitivity of the notion of quasi-unitary equivalence, we conclude:

Theorem 2 *Let $a \in \mathcal{H}$ be real valued and a_m its projection onto $\mathcal{H}_m$. Then $\mathcal{E}^a$ and $\mathcal{E}^{a_m}_m$ are $\widetilde{\delta}$-quasi-unitarily equivalent.*

Note that we tacitly assumed that $\mathcal{E}^a$ is closed in $\mathscr{H}_\infty$. This is e.g. true if the magnetic field is small enough (cf. [4] for more details).