Wavelet analysis of a multifractional process in an arbitrary Wiener chaos

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1. Introduction and motivations

Fractional Brownian motion (fBm) of an arbitrary Hurst parameter $H \in (0,1)$, denoted by $\{B_H(t) : t \in \mathbb{R}\}$, is defined, up to a multiplicative constant, as the unique (in distribution) Gaussian process with **stationary increments** which is **globally self-similar** of order H.

The representation of fBm as a well-balanced moving average is given, for every $t \in \mathbb{R}$, by the Wiener integral over $\mathbb{R}$:

$$B_H(t) = \int_{\mathbb{R}} \Big[|t-s|^{H-1/2} - |s|^{H-1/2} \Big] dB(s), \tag{1.1}$$

with the convention that $|t-s|^0 - |s|^0 = \log|t-s| - \log|s|$. FBm was first introduced by Kolmogorov in 1940 as a way for generating Gaussian spirals in Hilbert spaces. Later, in 1968, the well-known article by Mandelbrot and Van Ness emphasised its importance as a model in several areas of application: hydrology, geology, finance, and so on. Since then many applied and theoretical aspects of this stochastic process have been extensively explored in the literature and, among many other things, its sample behavior has been well understood. Despite its importance in modeling, fBm does not always succeed in giving a sufficiently reliable description of real-life signals. Indeed, fBm suffers from **two main limitations**:

(a) its Gaussian character,

(b) local roughness of its sample paths remains everywhere the same; more precisely, their local and **pointwise Hölder exponents** are everywhere equal to the Hurst parameter H.

In this work, we construct a **natural extension** of FBM denoted by Z, which belongs to a **homogeneous Wiener chaos** of an arbitrary order $d \in \mathbb{Z}_+$ and whose local regularity changes from one point to another. Then, using a **wavelet approach**, we study its global and local behavior. Namely, the chaotic multifractional process Z of functional paramater $H(\cdot)$ is defined, for all $t \in \mathbb{R}$, through the multiple Wiener integral on $\mathbb{R}^d$:

$$Z(t) = \int_{\mathbb{R}^d} \Big[\|\mathbf{t}^* - \mathbf{x}\|_2^{H(t)-\frac{d}{2}} - \|\mathbf{x}\|_2^{H(t)-\frac{d}{2}} \Big] dB_{x_1} \ldots dB_{x_d}, \tag{1.2}$$

where $\mathbf{t}^* = (t, \ldots, t) \in \mathbb{R}^d$, $\|\cdot\|_2$ denotes the Euclidian norm over $\mathbb{R}^d$, and $H(\cdot)$ is an arbitrary deterministic continuous function over $\mathbb{R}$ with values in the open interval $(0,1)$.

2. Wavelet representation of the chaotic process

Let $E = \{0,1\}^d \backslash \{(0,\ldots,0)\}$. A Meyer wavelet basis of $L^2(\mathbb{R}^d)$ is an Hilbertian basis of $L^2(\mathbb{R}^d)$ of the form:

$$\Big\{ 2^{\frac{jd}{2}} \psi^{(\epsilon)}(2^j \mathbf{x} - \mathbf{k}) : j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^d, \epsilon \in E \Big\}; \tag{2.1}$$

for the sake of convenience, one sets:

$$\psi^{(\epsilon)}_{j,\mathbf{k}}(\mathbf{x}) = 2^{\frac{jd}{2}} \psi^{(\epsilon)}(2^j \mathbf{x} - \mathbf{k}). \tag{2.2}$$

The $2^d - 1$ real-valued functions $\psi^{(\epsilon)}$, $\epsilon \in E$, which generate the basis are called **the d-variate Meyer mother wavelets**. They can be expressed as tensor products of ψ^0 and ψ^1 which respectively denote a 1-variate Meyer father and mother wavelet. Using the nice properties of the d-variate Meyer mother wavelets, for each $\epsilon \in E$, it can be shown that the real-valued function Ψ^ϵ defined, for all $(\mathbf{u}, v) \in \mathbb{R}^d \times [0,1]$, as

$$\Psi^\epsilon(\mathbf{u}, v) = \int_{\mathbb{R}^d} \|\mathbf{u} - \mathbf{s}\|_2^{v-d/2} \psi^{(\epsilon)}(\mathbf{s}) d\mathbf{s},$$

is infinitely differentiable on $\mathbb{R}^d \times (0,1)$ and satisfies, as well as all its partial derivatives of any order, the following very useful **localization property** for all $(n, \mathbf{p}, q) \in \mathbb{Z}_+ \times \mathbb{Z}_+^d \times \mathbb{Z}_+$:

$$\sup \Big\{ \|\mathbf{u}\|_2^n \big| (\partial_{\mathbf{u}}^{\mathbf{p}} \partial_v^q \Psi^\epsilon)(\mathbf{u}, v) \big| : (\mathbf{u}, v) \in \mathbb{R}^d \times (0,1) \Big\} < +\infty. \tag{2.3}$$

Recall that a centred non-Gaussian square integrable real-valued random variable, on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, belongs to the **homogeneous Wiener chaos** of an arbitrary integer order $d \geq 2$ when it can be represented by a **multiple Wiener integral** over $\mathbb{R}^d$. We always denote by $I_d(\cdot)$ this stochastic integral, and use the classical convention that, for every $f \in L^2(\mathbb{R}^d)$, one has $I_d(f) = I_d(\tilde{f})$; the function $\tilde{f}$ being the symmetrization of f, defined, for all $(t_1, \ldots, t_d) \in \mathbb{R}^d$, as

$$\tilde{f}(t_1, \ldots, t_d) = \frac{1}{d!} \sum_{\sigma \in \mathcal{S}_d} f(t_{\sigma(1)}, \ldots, f(t_{\sigma(d)})),$$

where $\mathcal{S}_d$ refers to the set of all permutations of $\{1, \ldots, d\}$. A very important property of multiple Wiener integrals, which somehow can be viewed as an **isometry property**, is that, for all function $f \in L^2(\mathbb{R}^d)$, one has

$$\mathbb{E}\big(|I_d(f)|^2 \big) = d! \, \|\tilde{f}\|^2_{L^2(\mathbb{R}^d)} \leq d! \, \|f\|^2_{L^2(\mathbb{R}^d)}. \tag{2.4}$$

By expanding, for each fixed $t \in \mathbb{R}$, the kernel function $\mathbf{x} \mapsto \|\mathbf{t}^* - \mathbf{x}\|_2^{H(t)-\frac{d}{2}} - \|\mathbf{x}\|_2^{H(t)-\frac{d}{2}}$ in (1.2) into a Meyer wavelet basis of $L^2(\mathbb{R}^d)$ (see e.g [4]), and by using the isometry property of the multiple Wiener integral, we construct a random series representation for the chaotic fractional process $\{Z(t) : t \in \mathbb{R}\}$ which converges in $L^2(\Omega)$ where Ω denotes the underlying probability space. The following proposition provides the random wevelts series and improves its type of convergence.

Proposition 2.1 *For each fixed $t \in \mathbb{R}$, one has*

$$Z(t) = \sum_{j,\mathbf{k},\epsilon} 2^{-jH(t)} \big(\Psi^\epsilon(2^j \mathbf{t}^* - \mathbf{k}, H(t)) - \Psi^\epsilon(-\mathbf{k}, H(t)) \big) I_d(\psi^{(\epsilon)}_{j,\mathbf{k}})(\omega),$$

where the random series in the right-hand side is, on an event Ω^* of probability 1 , uniformly convergent in t, on each compact subset of $\mathbb{R}$.

The proof of Proposition 2.1 is mainly based on the localization property (2.3) and on the following important lemma borrowed from [1].

Lemma 2.2 *For each $(j, \mathbf{k}, \epsilon) \in \mathbb{Z} \times \mathbb{Z}^d \times E$, let $I_d(\psi^{(\epsilon)}_{j,\mathbf{k}})$ be the multiple Wiener integral over $\mathbb{R}^d$ of the wavelet function defined in (2.2). That is one has*

$$I_d(\psi^{(\epsilon)}_{j,\mathbf{k}}) = \int_{\mathbb{R}^d} \psi^{(\epsilon)}_{j,\mathbf{k}}(\mathbf{x}) dB_{x_1} \ldots dB_{x_d}. \tag{2.5}$$

Then, there exists an event Ω^* of probability 1 and a finite positive random variable C_d such that, for all $\omega \in \Omega^*$ and for each $(j, \mathbf{k}, \epsilon) \in \mathbb{Z} \times \mathbb{Z}^d \times E$, one has

$$|I_d(\psi^{(\epsilon)}_{j,\mathbf{k}})(\omega)| \leq C_d(\omega) \big(\log(e + |j| + \|\mathbf{k}\|_1) \big)^{\frac{d}{2}}, \tag{2.6}$$

where $\|\cdot\|_1$ denotes the 1-norm over $\mathbb{R}^d$.

3. Global and local behavior

Using the representation of the process Z as a random wevelet series, we obtain almost surely, **global** and **local modulus of continuity** as well as a log-iterated law of its paths.

Theorem 3.1 *Let $H(\cdot)$ be the continuous functional parameter of the chaotic multifractional process $\{Z(t) : t \in \mathbb{R}\}$. Let $\mathcal{K} \subset \mathbb{R}$ be an arbitrary non degenerate compact interval. One sets*

$$\underline{H}(\mathcal{K}) := \min\{H(t) : t \in \mathcal{K}\}$$

Assuming that

$$H(\cdot) \in C^{\gamma_{\mathcal{K}}}(\mathcal{K}) \quad \text{for some} \quad \gamma_{\mathcal{K}} \in [\underline{H}(\mathcal{K}), 1), \tag{3.1}$$

where $C^{\gamma_{\mathcal{K}}}(\mathcal{K})$ denotes the global space of Hölder on $\mathcal{K}$ of order $\gamma_{\mathcal{K}}$. Then, for all $\omega \in \Omega^*$, one has:

$$\sup_{(t_1,t_2) \in \mathcal{K}^2} \left\{ \frac{|Z(t_1, \omega) - Z(t_2, \omega)|}{|t_1 - t_2|^{\underline{H}(\mathcal{K})} \Big(1 + \big| \log|t_1 - t_2| \big| \Big)^{\frac{d}{2}}} \right\} < +\infty. \tag{3.2}$$

Theorem 3.2 *Let $t_0 \in \mathbb{R}$ be an arbitrary fixed point. Assume that there exists a constant $\gamma_{t_0} \in [H(t_0), 1)$ such that the continuous function $H(\cdot)$ satisfies*

$$\sup_{t \in \mathbb{R}} \left\{ \frac{|H(t) - H(t_0)|}{|t - t_0|^{\gamma_{t_0}}} \right\} < +\infty. \tag{3.3}$$

Then, one has, almost surely:

$$\sup_{t \in [t_0-1, t_0+1]} \left\{ \frac{|Z(t) - Z(t_0)|}{|t - t_0|^{H(t_0)} \Big(\log \Big(e + \big| \log|t - t_0| \big| \Big) \Big)^{\frac{d}{2}}} \right\} < +\infty. \tag{3.4}$$

The following theorem shows that the chaotic multifractional process $\{Z(t) : t \in \mathbb{R}\}$ has **a local asymptotic self-similarity** property rather similar to the one satisfied by the classical Gaussian multifractional Brownian motion (see [3]).

Theorem 3.3 *Let $t_0 \in \mathbb{R}$ be an arbitrary fixed point such that the condition (3.3) holds. Then, the stochastic process $\{Z(t) : t \in \mathbb{R}\}$ is at t_0, **strongly locally asymptotically self-similar** of order $H(t_0)$ and the **tangent process** is $\{X(s, H(t_0)) : s \in \mathbb{R}\}$. More precisely, let $(\nu_n)_{n \in \mathbb{N}}$ be an arbitrary sequence of positive real numbers which converges to 0. For each $n \in \mathbb{N}$, let $T_{t_0,\nu_n} Z = \{(T_{t_0,\nu_n} Z)(s) : s \in \mathbb{R}\}$ be the stochastic process with continuous paths, defined, for all $s \in \mathbb{R}$, as*

$$(T_{t_0,\nu_n} Z)(s) = \frac{Z(t_0 + \nu_n s) - Z(t_0)}{\nu_n^{H(t_0)}}. \tag{3.5}$$

Then, when n goes to $+\infty$, the probability measure induced on $\mathcal{C}(\mathcal{J})$ by $\{(T_{t_0,\nu_n} Z)(s) : s \in \mathbb{R}\}$ converges to the one induced on $\mathcal{C}(\mathcal{J})$ by $\{X(s, H(t_0)) : s \in \mathbb{R}\}$, where $\mathcal{C}(\mathcal{J})$ denotes the usual Banach space of the real-valued continuous functions over an arbitrary non degenerate compact interval $\mathcal{J}$ of the real line equipped with the uniform norm.

Remark 3.4 One can derive from Theorem 3.3 and zero-one law that, for any fixed arbitrarily small positive real number η, one has, almost surely,

$$\sup_{t \in [t_0-1, t_0+1]} \left\{ \frac{|Z(t) - Z(t_0)|}{|t - t_0|^{H(t_0)+\eta}} \right\} = +\infty,$$

which means that the exponent $H(t_0)$ in (3.4) is optimal. Moreover, when $\gamma_{\mathcal{K}}$ in (3.1) belongs to $[\overline{H}(\mathcal{K}), 1)$, then, using similar arguments, it can be shown that the exponent $\underline{H}(\mathcal{K})$ in (3.2) is optimal: one has, almost surely,

$$\sup_{(t_1,t_2) \in \mathcal{K}^2} \left\{ \frac{|Z(t_1) - Z(t_2)|}{|t_1 - t_2|^{\underline{H}(\mathcal{K})+\eta}} \right\} = +\infty.$$

Theorem 3.5 *Assume that the continuous function $H(\cdot)$ is with values in a compact interval included in $(0,1)$ (this means that $\inf_{t \in \mathbb{R}} H(t) > 0$ and $\sup_{t \in \mathbb{R}} H(t) < 1$). Then, for each fixed $\omega \in \Omega^*$ and $\delta > 0$, one has:*

$$\sup_{|t| \geq \delta} \left\{ \frac{|Z(t, \omega)|}{|t|^{H(t)} \Big(\log \Big(e + \big| \log|t| \big| \Big) \Big)^{\frac{d}{2}}} \right\} < +\infty. \tag{3.6}$$

References

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