General Position Theorem and its applications

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Motivation

Let K be the attractor of a system $\mathcal{S} = \{S_1, ...S_m\}$ of contraction similarities of $\mathbb{R}^n$. Suppose $S_i(K) \cap S_j(K) \neq \varnothing$ for some i, j. Under what conditions it is possible to change the system $\mathcal{S}$ slightly to such system $\mathcal{S}' = \{S'_1, ...S'_m\}$, that its attractor K' satisfies the condition $S'_i(K) \cap S'_j(K) = \varnothing$?

Our approach

General Position Theorem

Let (D, ρ), (L_1, σ_1), (L_2, σ_2) be metric spaces. Let $\varphi_1 : D \times L_1 \to \mathbb{R}^n$, $\varphi_2 : D \times L_2 \to \mathbb{R}^n$ be continuous maps such that:

(a) there exist $C > 0$ and $\alpha > 0$ such that for any $\xi \in D, x_1 \in L_1, x_2 \in L_2$ and for $i = 1, 2$:

$$\|\varphi_i(\xi, x_1) - \varphi_i(\xi, x_2)\| \le C[\sigma_i(x_1, x_2)]^\alpha;$$

(b) there exists $M > 0$ such that for any $(x_1, x_2) \in L_1 \times L_2$, $\xi, \xi' \in D$ the function

$$\Phi(\xi, x_1, x_2) := \varphi_1(\xi, x_1) - \varphi_2(\xi, x_2)$$

satisfies

$$\|\Phi(\xi', x_1, x_2) - \Phi(\xi, x_1, x_2)\| \ge M[\rho(\xi', \xi)].$$

Then the set

$$\Delta := \{\xi \in D : \varphi_1(\xi, L_1) \cap \varphi_2(\xi, L_2) \neq \varnothing\}$$

is closed in D and

$$\dim_H \Delta \le (1/\alpha) \dim_H(L_1 \times L_2).$$

Outline of the proof:
Consider implicit function $g : L_1 \times L_2 \to D$, such that $\Phi(g(x_1, x_2), x_1, x_2) = 0$.
(b) $\Rightarrow g$ exists;
(a)&(b) $\Rightarrow g$ is α-Hölder continuous. ∎

In compare to method, which uses transversality condition with potential-theoretic characterization of Hausdorff dimension [4], our method allows us to construct self-similar sets with prescribed behavior of critical set in general position, using transversality-like condition **(b)**.

Example 1. Twofold Cantor sets [1]

Relative position of the pieces of K.

$\mathcal{S}_{pq} = \{S_1, S_2, S_3, S_4\}$ in $\mathbb{R}$;
$S_1(x) = px$, $S_2(x) = qx$, $S_3(x) = px + 1 - p$, $S_4(x) = qx + 1 - q$; $\quad p, q \in (0, 1/16)$.
$K = K_{pq}$ is attractor of $\mathcal{S}_{pq}$, $\quad K_i = S_i(K_{pqr})$.

If the system $\mathcal{S}_{pq}$ satisfies the following exact overlap condition:

$$S_1^m(K) \cap S_2^n(K) = S_1^m S_2^n(K),$$

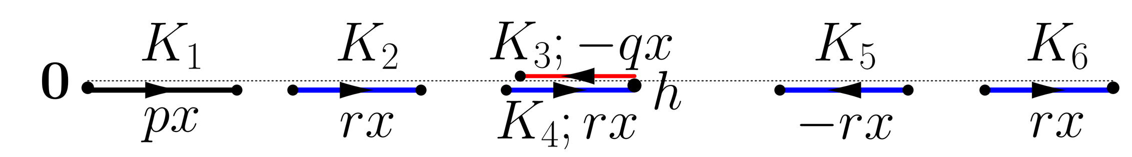
for all $m, n \in \mathbb{N}$, we call K_{pq} a *twofold Cantor set.*

Theorem 1

(1) Let $p \in (0, 1/16)$. Then K_{pq} is a twofold Cantor set for Lebesgue-almost all $q \in (0, 1/16)$.
(2) If K_{pq} is a twofold Cantor set, then $\mathcal{S}_{pq}$ does not have WSP.
(3) If K_{pq} is a twofold Cantor set, then $d = \dim_H K_{pq}$ satisfies the equation $p^d + q^d - (pq)^d = 1/2$.

Outline of the proof:
(1) Since $K = \{0\} \cup \bigcup\limits_{m,n=0}^{\infty} S_1^m S_2^n(A)$, where $A = S_3(K) \cup S_4(K)$, and this union is disjoint for twofold Cantor set - it is enough to prove that $S_1^m(A) \cap S_2^n(A) = \varnothing$ for all m, n. Fix $p \in (0, 1/16)$ and consider $\varphi_1 = S_1^m S_i \pi$, $\varphi_2 = S_2^n S_j \pi$, where $i, j \in \{3, 4\}$, $I = \{1, 2, 3, 4\}$ and $\pi : I^\infty \to K$ is a natural projection. Supply I^∞ with a metric such that φ_1, φ_2 are Lipschitz and apply General Position Theorem. Finally take a union over all m, n.
(2) Consider $S_1^m(S_2^n)^{-1}$ and use that $\dfrac{\log p}{\log q} \notin \mathbb{Q}$.
(3) Use a systems $\{S_1, S_1\omega, S_2\omega, \ldots, S_2^n\omega\}$ with $\omega(x) = 1 - x$ to get a lower estimates tending to $\dim_H K_{pq}$ as $n \to \infty$, and an infinite version of such system to get an upper estimate. ∎

Example 2. Even unique intersection point can break OSC [2]



Relative position of the pieces of K.

$\mathcal{S}_{pqr} = \{S_1, S_2, ..., S_6\}$ in $\mathbb{R}$;
$S_1(x) = px, S_2(x) = a + rx, S_3(x) = h - qx,$
$S_4(x) = h - r + rx, S_5(x) = 1 - a - rx,$
$S_6(x) = 1 - r + rx; \qquad h = 8/15, \ a = 3/15,$
$p, q, r \in (0, 1/36).$
$K = K_{pqr}$ is attractor of $\mathcal{S}_{pqr}$, $\quad K_i = S_i(K_{pqr})$.

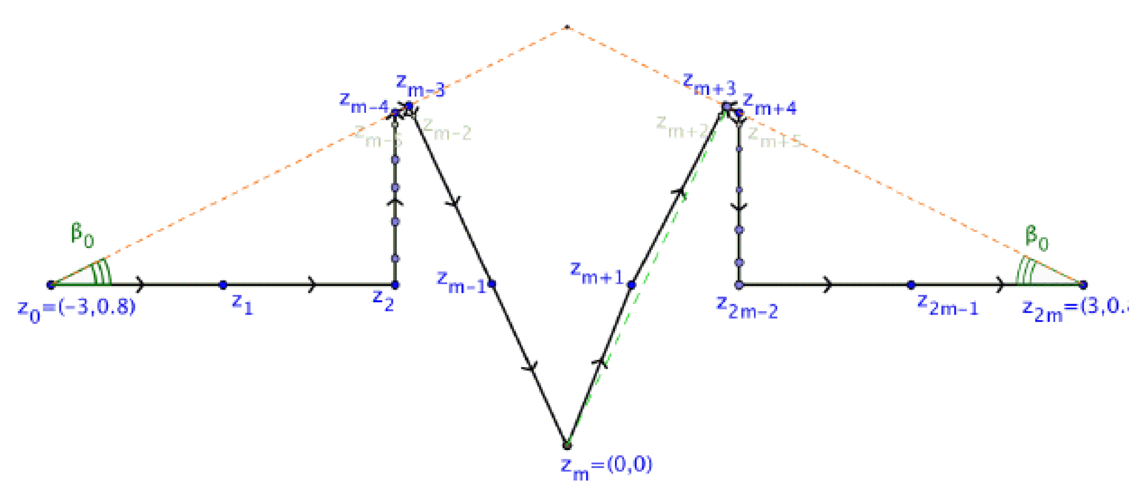
By the construction, the only possible non-empty intersection of the pieces is $K_3 \cap K_4$. In the case $K_3 \cap K_4 = \{h\}$, we say that the system $\mathcal{S}_{pqr}$ has *unique one-point intersection.*

Theorem 2

(1) Fix $p, r \in (0, 1/36)$. Then for Lebesgue-almost all $q \in (0, 1/36)$ the system $\mathcal{S}_{pqr}$ has unique one-point intersection.
(2) If $\dfrac{\log p}{\log r} \notin \mathbb{Q}$, then the system $\mathcal{S}_{pqr}$ does not have WSP for any q.
(3) $\mathcal{S}_{pqr}$ has unique one-point intersection, then $\dim K_{pqr}$ coincides with similarity dimension of the system $\mathcal{S}_{pqr}$.

Outline of the proof:
(1) Analogous to that of Example 1. Use that $K = \{0\} \cup \bigcup\limits_{m=0}^{\infty} S_1^m(K \setminus K_1) = \{1\} \cup \bigcup\limits_{n=0}^{\infty} S_4^n(K \setminus K_6)$, and apply General Position Theorem to $\varphi = S_3 S_1^m S_i \pi, \psi = S_4 S_6^n S_j \pi$, where $i \in I \setminus \{1\}$, $j \in I \setminus \{6\}$, $I = \{1, \ldots, 6\}$ and $\pi : I^\infty \to K$ is a natural projection.
(2) Consider $(S_4 S_6^n S_2)^{-1} S_3 S_1^m S_5$.
(3) Use a systems $\{S_1^k S_j : k \in \{0, 1, \ldots, n\}, \ j \in I \setminus \{1\}\}$ to get a lower estimates tending to $\dim_H K_{pq}$ as $n \to \infty$. ∎

Example 3. Self-similar Jordan curve in $\mathbb{R}^3$, not satisfying WSP [3]



Theorem 3

There is such system $\mathcal{S} = \{S_1, ..., S_m\}$ of contraction similarities in $\mathbb{R}^3$, which:
(1) does not satisfy WSP,
(2) satisfies one-point intersection property,
(3) whose attractor is a Jordan arc.

Questions answered

Does finite intersection property imply...

- OSC, or at least WSP? - No and no.
- positive Hausdorff measure? - No.
- WSP for connected self-similar sets in $\mathbb{R}^3$? - No.

References

1. K. Kamalutdinov, A. Tetenov, *Even unique intersection point can break OSC: an example*, arXiv:1809.08595.
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3. A. Tetenov, K. Kamalutdinov, D. Vaulin, *Self-similar Jordan arcs which do not satisfy OSC*, arXiv:1512.00290.
4. K. Simon, B. Solomyak, M Urbański, *Hausdorff dimension of limit sets for parabolic IFS with overlaps*, Pacific J. Math. 201:2 (2001), pp. 441–478.

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