Sobolev spaces and calculus of variations on fractals

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Michael Hinz, Dorina Koch, Melissa Meinert
Bielefeld University, Dept. of Mathematics
mhinz@math.uni-bielefeld.de, dorina-koch@t-online.de, mmeinert@math.uni-bielefeld.de

Bielefeld Seoul
International Research Training Group 2235

Introduction

We pursue two aims.

1. Quick account on p-energies and $(1,p)$-Sobolev spaces for fractals that carry a local regular Dirichlet form.
2. Adapt a classical result [6, Theorem 4.3.1] about the existence of minimizers for convex functionals in the present setup.

Materials and Methods

Dirichlet forms, p-energies and Sobolev spaces $H_0^{1,p}(X,m)$

- (X,d): locally compact separable metric space,
- m: nonneg. Radon measure on X s.t. $m(U)>0$ for any nonempty open set $U\subset X$,
- $(\mathcal{E},\mathcal{D}(\mathcal{E}))$: given regular Dirichlet form on $L^2(X,m)$.

Idea: Partially generalize former definitions in [5, Section 6], which covered the cases $2\le p<+\infty$.

We make the following *standing assumptions*.

Assumption 1. The Dirichlet form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ on $L^2(X,m)$ is strongly local and admits a *carré du champ*, $m(X)<+\infty$, and $\mathcal{A}$ is an algebra and a core for $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ s.t.

$$\Gamma(f,g):=\frac{d\Gamma(f,g)}{dm}\in L^\infty(X,m)\quad \forall f,g\in\mathcal{A}.$$

Assumption 2. There is a space $\mathcal{A}_\mathcal{L}\subset\mathcal{D}(\mathcal{L})$, dense in $\mathcal{D}(\mathcal{E})$ and s.t. for any $f\in\mathcal{A}_\mathcal{L}$ we have $\Gamma(f)\in L^\infty(X,m)$ and $\mathcal{L}f\in L^\infty(X,m)$.

Remark 1. As a by-product one can provide an analog of the most classical definition of Sobolev spaces $W^{1,p}(\Omega)$.

- Define associated p-*energies*, $1\le p<+\infty$, by $\mathcal{E}^{(p)}(f):=\int_X\Gamma(f)^{p/2}dm,\quad f\in\mathcal{A}$.

Theorem 1. *The functional $(\mathcal{E}^{(p)},\mathcal{A})$ is closable in $L^p(X,m)$, $2\le p<+\infty$. If Assumption 2 is satisfied, then it is also closable in $L^p(X,m)$, $1<p<2$.*

- Suppose $f\in L^p(X,m)$ s.t. there exists a sequence $(f_n)_n\subset\mathcal{A}$, Cauchy in the seminorm $\mathcal{E}^{(p)}(\cdot)^{1/p}$ and convergent to f in $L^p(X,m)$ and define $\mathcal{E}^{(p)}(f):=\lim_n\mathcal{E}^{(p)}(f_n)$.
- Denote the vector space of all such $f\in L^p(X,m)$ by $H_0^{1,p}(X,m)$. $H_0^{1,p}(X,m)$ are Banach with norms
$$\|f\|_{H_0^{1,p}(X,m)}=\|f\|_{L^p(X,m)}+\mathcal{E}(f)^{1/p},\quad f\in H_0^{1,p}(X,m).$$

Similar to [3]

Definition 1. *To the spaces $H_0^{1,p}(X,m)$, $1\le p<+\infty$, we refer as Sobolev spaces. Given an open set $\Omega\subset X$ we define $H_0^{1,p}(\Omega,m)$ on Ω as the completion in $H_0^{1,p}(X,m)$ of all elements of $\mathcal{A}$ supported in Ω, respectively.*

L^p-vector fields and reflexivity of Sobolev spaces

- Following [2], one can
  - construct a Hilbert space $(\mathcal{H},\langle\cdot,\cdot\rangle_\mathcal{H})$ s.t. for all $a,b,c,d\in C_c(X)\cap\mathcal{D}(\mathcal{E})$ we have $a\otimes b, c\otimes d\in\mathcal{H}$ and
  $$\langle a\otimes b,c\otimes d\rangle_\mathcal{H}=\int_X bd\Gamma(a,c)dm.$$
  - introduce a derivation $\partial f:=f\otimes\mathbf{1}$, $f\in\mathcal{A}$, and extend it to a closed unbounded linear operator $\partial:L^2(X,m)\to\mathcal{H}$ with domain $\mathcal{D}(\mathcal{E})$, and $\|\partial f\|_\mathcal{H}^2=\mathcal{E}(f)$, $f\in\mathcal{D}(\mathcal{E})$.
  - refer to $\mathcal{H}$ as the *space of generalized L^2-vector fields*.

In particular, there exists a measurable field $(\mathcal{H}_x)_{x\in X}$ of Hilbert spaces (see [8]) s.t.
$\langle u,v\rangle_\mathcal{H}=\int_X^\oplus\langle u_x,v_x\rangle_{\mathcal{H}_x}\,m(dx)\quad\forall u,v\in\mathcal{H}$.
For $v=(v_x)_{x\in X}$ let

$$\|v\|_{L^p(X,m,(\mathcal{H}_x)_{x\in X})}:=\left(\int_X\|v_x\|_{\mathcal{H}_x}^p m(dx)\right)^{\frac{1}{p}},\qquad 1\le p<\infty,$$

and define the spaces $L^p(X,m,(\mathcal{H}_x)_{x\in X})$ as the collections of the respective equivalence classes of m-a.e. equal sections having finite norm.

Similar to [1]

Proposition 1. *The spaces $L^p(X,m,(\mathcal{H}_x)_{x\in X})$, $1<p<+\infty$, are uniformly convex and in particular, reflexive. For each $1<p<+\infty$ the spaces $L^p(X,m,(\mathcal{H}_x)_{x\in X})$ and $L^q(X,m,(\mathcal{H}_x)_{x\in X})$ with $1=1/p+1/q$ are the dual of each other.*

Main Result

Theorem 2. *([6, Theorem 4.3.1]) Let $1<p<+\infty$ let $\Omega\subset X$ be an open set and assume that the Poincaré inequality*
$$\|u\|^p_{L^p(\Omega,m)}\le c\,\mathcal{E}^{(p)}(u),\quad u\in H_0^{1,p}(\Omega,m),$$
holds, where $c>0$ is constant depending only on Ω and p. Let $f=(f_x)_{x\in X}$ be a family of mappings $f_x:\mathcal{H}_x\to\mathbb{R}$, $x\in X$ s.t.

(i) for every $v\in L^p(X,m,(\mathcal{H}_x)_{x\in X})$ the function $x\mapsto f_x(v_x)$ is Borel measurable,

(ii) the function f_x is lower semicontinuous and convex for all $x\in X$,

(iii) there are a function $a\in L^1(X,m)$ and constant $b>0$ is satisfied s.t.
$$f_x(v_x)\ge -a(x)+b\|v_x\|^p_{\mathcal{H}_x}$$
for a.a. $x\in X$ and all $v\in L^p(X,m,(\mathcal{H}_x)_{x\in X})$.

Then for any $g\in H_0^{1,p}(X,m)$ the functional $I[u]=\int_X f_x(\partial_x u)m(dx)$ admits its infimum on $g+H_0^{1,p}(\Omega,m)$.

Examples

Degenerate forms
Let $X=(-1,1)^2\subset\mathbb{R}^2$ and consider the quadratic form

$$\mathcal{E}(f)=\int_{-1}^1\int_{-1}^1\left(\frac{\partial f}{\partial x_1}\right)^2dx_1dx_2+\int_{-1}^1\int_0^1 x_2\left(\frac{\partial f}{\partial x_2}\right)^2dx_1dx_2,\quad f\in C_c^\infty((-1,1)^2).$$

Since $\frac{\partial}{\partial x_i}(x_2\vee 0)\in L^2((-1,1)^2)$, $i=1,2$, the form is closable in $L^2((-1,1)^2)$, [4, Section 3.1, (1°.a)], and its closure satisfies Assumptions 1 and 2 with m being the two-dimensional Lebesgue measure, $dm=dx_1dx_2$ and $\mathcal{A}_\mathcal{L}=\mathcal{A}=C_c^\infty(\Omega)$.

Here: energy functional with 'varying tangent space dimensions'.

Sierpinski gasket
Let X be the class. Sierpinski gasket K and $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ its standard energy form, see for instance [7]. Consider it in $L^2(K,\nu)$, where $m=\nu$ is the Kusuoka measure, $\nu:=\nu_{h_1}+\nu_{h_2}$, and $\{h_1,h_2\}$ is an energy orthonormal system of non-constant harmonic functions on K.

Assumptions 1 and 2 are satisfied, this follows from results in [9].

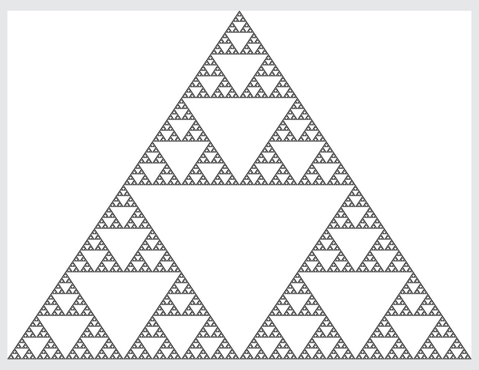

Figure 1: Sierpinski gasket

Anisotropic functionals
Let $1<p<+\infty$, if $1<p<2$ let Assumption 2 be in force. Suppose that for m-a.e. $x\in X$ the space $\mathcal{H}_x$ is two-dimensional. Let $\eta^{(1)},\eta^{(2)}\in\mathcal{H}$ be s.t. for any $x\in X$ with $\dim\mathcal{H}_x=2$, $\left\{\eta_x^{(1)},\eta_x^{(2)}\right\}$ is an orthonormal basis in $\mathcal{H}_x$, see for instance [8, Lemma 8.12]. By Theorem 2 we can find a minimizer in $g+H_0^{1,p}(\Omega,m)$ for the functional I with integrand defined by

$$f_x(v)=\|v\|^p_{\mathcal{H}_x}+\left|\left\langle v,\eta_x^{(1)}\right\rangle_{\mathcal{H}_x}\right|^p,\quad v\in\mathcal{H}_x,$$

if $\mathcal{H}_x$ is two-dimensional and by $f_x\equiv 0$ otherwise. This anisotropic functional could not be expressed in terms of the carré operator $u\mapsto\Gamma(u)$ only.

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