Augmented trees for infinite IFSs

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Abstract

We generalize the notion of augmented tree [LW17] to infinite IFS of similarity mappings on $\mathbb{R}^d$. This graph Γ is δ-hyperbolic in the sense of Gromov and allows us to define the hyperbolic boundary. The space $\partial^{g}\Gamma$ of equivalence classes of geodesic rays is a subset of the hyperbolic boundary. The usual coding map adapted to the notion of augmented tree defines a Hölder equivalence of between $\partial^{g}\Gamma$ and the limit set of the IFS.

Notation

Let $X \subset \mathbb{R}^d$ be a compact set and wlog $|X| = 1$. For an infinite index set $I \subset \mathbb{N}$, consider an IFS Φ of contractive similarity mappings $\varphi_i : X \to X$, $i \in I$. Denote by R_i the contraction ratio of φ_i, $i \in I$, and assume wlog $\lim_{i\in I} R_i = 0$ and wlog $R_i \le R_j$ for $i \le j$.

- I^n: words of n letters from alphabet I, $n \in \mathbb{N}$; $\quad I^0 \coloneqq \{()\}$, where $()$ is the empty word
- $I^* \coloneqq \bigcup\{I^n | n \in \mathbb{N}_0\}$
- $I^{\mathbb{N}}$: sequences of letters from alphabet I; for $\xi \in I^{\mathbb{N}}$, $\xi|n$: the prefix of ξ of n letters, $n \in \mathbb{N}$

For $\omega = (\omega_1, \dots, \omega_n) \in I^n$, $n \in \mathbb{N}_0$, let

- if $n = 0$: $\varphi_{()} \coloneqq \mathrm{Id}_{\mathbb{R}^d}$, else: $\varphi_\omega \coloneqq \varphi_{\omega_1} \circ \cdots \circ \varphi_{\omega_n}$ with ratio R_ω
- for $\tau = \tau_1 \dots \tau_m \in I^m$, $m \in \mathbb{N}_0$: $\omega\tau \coloneqq \omega_1 \dots \omega_n \tau_1 \dots \tau_m \in I^{n+m}$
- $\omega^- \coloneqq (\omega_1, \dots, \omega_{n-1})$

The *limit set* of the IFS is $J = \bigcup_{\xi \in I^{\mathbb{N}}} \bigcap_{n=1}^{\infty} \varphi_{\xi|n}(X)$. It is not closed in general.
Following the ideas in [LW17] we construct a graph, an *augmented tree*, associated to the IFS Φ.

Augmented tree $(\Gamma^*, \mathcal{E})$

Let $R = R_1$ and consider the subsets $F_k \subset I^*$ for $k \in \mathbb{N} \cup \{0\} \eqqcolon \mathbb{N}_0$ given by

$$F_0 \coloneqq \{()\}, \qquad F_k \coloneqq \left\{\omega \in I^* \,\middle|\, R_\omega \le R^k < R_{\omega^-}\right\}, \qquad k \in \mathbb{N}. \qquad \Rightarrow \qquad \bigcup\{F_k | k \in \mathbb{N}_o\} = I^*$$

Note that contrary to the construction in [LW17], the sets F_k, $k \in \mathbb{N}_0$, are not disjoint. For each $k \in \mathbb{N}_0$, we interpret the sets F_k as level sets of vertices Γ_k via function

$$\mathsf{w} : \Gamma_k \to F_k, \quad \mathsf{w}(x) = \omega \quad \text{if } x \text{ is corresponds to the word } \omega$$

and extend it to $\mathsf{w} : \Gamma^* \coloneqq \bigcup \Gamma_k \to I^*$, on *vertex set* Γ^*.

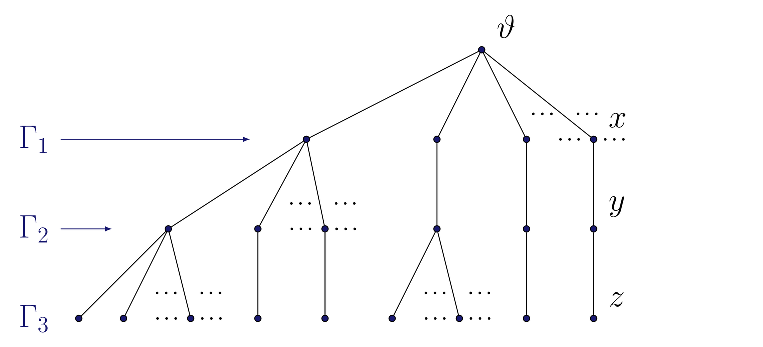

Figure 1: Example of vertical structure of the graph Γ. For the vertices x, y and z we have $\mathsf{w}(x) = \mathsf{w}(y) = \mathsf{w}(z)$.

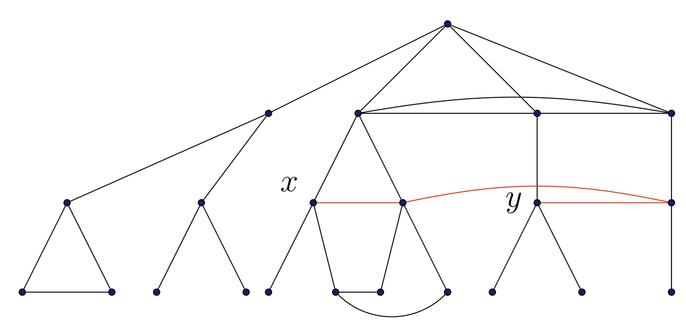

Figure 2: Example of finite subgraph of Γ with vertical and horizontal edges. The path from x to y is a horizontal geodesic.

Definition of augmented tree Γ

The *vertex set* is Γ^*. Let ϑ with $\mathsf{w}(\vartheta) = ()$, be the *root*. Fix $\kappa > 0$.

- *vertical (unorientated) edges* $\mathcal{E}_v$

$$(x, y) \in \mathcal{E}_v \quad \Leftrightarrow \quad x \in \Gamma_k \wedge y \in \Gamma_{k+1} \text{ for some } k \in \mathbb{N}_0 \;\wedge\; \mathsf{w}(y) = \mathsf{w}(x)\zeta \text{ for some } \zeta \in I^*$$

- *horizontal (unoriented) edges* $\mathcal{E}_h$:

$$(x, y) \in \mathcal{E}_h \quad \Leftrightarrow \quad x, y \in \Gamma^k \text{ for some } k \in \mathbb{N}, x \neq y \;\wedge\; \mathrm{dist}\left(\varphi_{\mathsf{w}(x)}(X), \varphi_{\mathsf{w}(y)}(X)\right) \le \kappa R^k.$$

- *augmented tree*: $\Gamma = (\Gamma^*, \mathcal{E})$ with set of *edges*: $\mathcal{E} \coloneqq \mathcal{E}_v \cup \mathcal{E}_h$
- (integer valued) *graph metric* on Γ^*: d
  path $[x_1, x_2, \dots, x_l] \subset (\Gamma^*, \mathcal{E})$ of length l is a geodesic $\Leftrightarrow d(x_1, x_l) = l$
  infinite path $[x_1, x_2, \dots]$ is *geodesic ray* $\Leftrightarrow d(x_k, x_l) = |l - k|$ for all $k, l \in \mathbb{N}$
  $\Pi \coloneqq$ set of all geodesic rays starting at ϑ
- for $k, l \in \mathbb{N}_0$ and $y \in \Gamma_{k+l}$: $y^{-l} = x$ if $x \in \Gamma_k$ and $\exists$ path $\subset \mathcal{E}_v$ connecting x and y

Properties of augmented tree Γ

The graph $(\Gamma^*, \mathcal{E}_v)$ is a tree with root ϑ and with vertices of infinite degree.
In particular, we have $I \equiv \Gamma_1$ and $\deg \vartheta = \infty$.

- $\forall k \in \mathbb{N} \forall x \in \Gamma_k \,\forall 0 \le l \le k : \varphi_{\mathsf{w}(x)}(X) \subset \varphi_{\mathsf{w}(x^{-l})}(X)$
- If $x, y \in \Gamma_k$ for some $k > 0$ and $(x, y) \in \mathcal{E}$ then either $x^{-1} = y^{-1}$ or $(x^{-1}, y^{-1}) \in \mathcal{E}_h$.
- $\forall \xi \in I^{\mathbb{N}} \,\forall k \in \mathbb{N} : \exists n(k) \in \mathbb{N} \,\exists x \in \Gamma_k : \xi|n(k) = \mathsf{w}(x)$
- $\forall \xi \in I^{\mathbb{N}} : \bigcap_{n=1}^{\infty} \varphi_{\xi|n}(X) = \bigcap_{k=1}^{\infty} \varphi_{\mathsf{w}(x_k)}(X)$ where x_k such that $\mathsf{w}(x_k) = \xi|n(k)$, $n(k)$ as above

Proposition

There exists $\gamma < \infty$ such that every horizontal geodesic in (Γ, d) is bounded in length by γ.
Hence, if $\pi = [\vartheta, x_2, x_3, \dots]$ is geodesic ray then $\pi \subset (\Gamma^*, \mathcal{E}_v)$. Furthermore, for $\xi \in I^{\mathbb{N}}$ and $(x_k)_k \in \Gamma^1 \times \Gamma^2 \times \dots$ with $\mathsf{w}(x_k) = \xi|n(k)$, the path $[\vartheta, x_1, x_2, \dots]$ is a geodesic ray.

δ-hyperbolicity

Following [Woe00], we recall the notions of δ-hyperbolic spaces and δ-Gromov hyperbolic graphs.

δ-hyperbolic space

Let (Γ, ρ) be a geodesic space. For $x, y \in \Gamma$ write $\pi[x, y]$ for a geodesic path from x to y, which in general is not unique. The space (X, d) is δ-hyperbolic for $\delta \ge 0$ if the following holds. For any $x, y, z \in \Gamma$ consider triangle composed by geodesic paths $\pi[x, y], \pi[y, z], \pi[z, x]$. Then any point on one of the three sides is at distance of at most δ from some point on one of the other sides.

Gromov product & δ-Gromov hyperbolic graphs

Let (Γ, d) be a graph with graph metric d and ϑ a vertex in Γ. For vertices x, y in Γ define *Gromov product of x and y wrt reference point ϑ* as

$$|x \wedge y| \coloneqq \frac{1}{2}(d(x, \vartheta) + d(y, \vartheta) - d(x, y)).$$

The graph (Γ, d) with graph metric d is *δ-Gromov hyperbolic* if for some choice (and hence any choice with some other δ) of reference point there exists $\delta \ge 0$

$$|x \wedge y| \ge \min\{|x \wedge z|, |y \wedge z|\} - \delta, \quad x, y, z \in \Gamma.$$

Consider a graph (Γ, d) and extend the graph metric d in canonical way to edges. Then

(Γ, d) is δ-Gromov hyperbolic $\Rightarrow$ (Γ, d) is 4δ-hyperbolic
(Γ, d) is δ-hyperbolic space $\Rightarrow$ (Γ, d) is 3δ-Gromov hyperbolic

The graph $(\Gamma^*, \mathcal{E})$ is δ-hyperbolic

We now return to the graph $\Gamma = (\Gamma^*, \mathcal{E})$ from above.

Proposition

Let (Γ, d) be the augmented tree for fixed $\kappa > 0$ with the usual graph metric d.

- Γ is δ-hyperbolic $\Leftrightarrow$ every horizontal geodesic is bounded in length.
- Hence, (Γ, d) is δ-hyperbolic space for some $\delta \ge 0$.

Hyperbolic ultrametric ρ_a & hyperbolic boundary of Γ

For $\delta \ge 0$ such that Γ is δ-hyperbolic, let $a > 0$ such that $\exp(3\delta a) < \sqrt{2}$.

- Define ultrametric ρ_a wrt reference point $\vartheta = ()$ of $x, y \in \Gamma^*$:

$$\rho_a(x, y) \coloneqq \begin{cases} \exp(-a|x \wedge y|), & x, y \in \Gamma^*, x \neq y \\ 0, & x = y. \end{cases}$$

- There exists a metric θ_a equivalent to ρ_a.
- Consider the completion $\widehat{\Gamma}$ of (Γ, θ_a). The *hyperbolic boundary* is $\partial\Gamma \coloneqq \widehat{\Gamma} \setminus \Gamma$.

Proposition

- Let $(x_n)_n$ be a sequence in Γ^* with $d(x_n, \vartheta) \to \infty$.
  $(x_n)_n$: Cauchy sequence wrt $\theta_a \Leftrightarrow (x_n)_n$: Cauchy sequence wrt $\rho_a \Leftrightarrow \lim_{m,n\to\infty} |x_n \wedge x_m| = \infty$
- Cauchy sequences $(x_n)_n$ and $(y_n)_n$ in (Γ^*, θ_a) define the same boundary point
  $\Leftrightarrow \lim_{n\to\infty} |x_n \wedge y_n| = \infty$.

Geodesic rays

Geodesic rays $\pi = [x_1, x_2, \dots]$, $\pi' = [y_1, y_2, \dots]$ in Γ are *equivalent* iff they stay at finite distance from each other, i. e. $\liminf_{n\to\infty} d(y_n, \pi) < \infty$. This relation is an equivalence relation on geodesic rays.

- If a geodesic ray $\pi = [x_1, x_2, ...]$ starts in a vertex $x_1 \neq \vartheta$ there exists a geodesic path $[\vartheta = z_1, \dots, z_l]$ with $l \in \mathbb{N}$, and $k \in \mathbb{N}_0$, such that $\tilde{\pi} \coloneqq [\vartheta, z_2, \dots, z_l, x_k, \dots]$ is a geodesic ray starting in ϑ.
- A geodesic ray $\tilde{\pi}$ constructed as above is equivalent to π.
- If $\pi = [x_1 = \vartheta, x_2, \dots]$ is a geodesic ray then $(x_n)_n$ is a Cauchy sequence.

Geodesic boundary $\partial^{g}\Gamma$

Let us define the *geodesic boundary* $\partial^{g}\Gamma$ as the quotient $\Pi/\sim$.

- In general, $\partial^{g}\Gamma \subseteq \partial\Gamma$.
- The Gromov product and the ultrametric ρ_a can be extended to ∂^{g} via
  $|[\pi] \wedge [\pi']| \coloneqq \inf\{\liminf_{n\to\infty} d(y_n, \pi) \,|\, \pi \in [\pi], \pi' = [\vartheta, y_1, y_2, \dots] \in [\pi']\}$.
- Define *coding map* $\mathsf{c} : \partial^{g}\Gamma \to J$ by

$$\mathsf{c}([\pi]) \coloneqq \bigcap_{k=1}^{\infty} \varphi_{\mathsf{w}(x_k)}(X), \quad \text{where } [\vartheta, x_1, x_2, \dots] \in [\pi].$$

Theorem

The coding map c is a well-defined bijection $\partial^{g}\Gamma \to J$.
The geodesic boundary $\partial^{g}\Gamma$ is Hölder equivalent to limit set J: $\exists C > 0$ such that, with $\alpha \coloneqq -1/a \ln R$,

$$C^{-1}|\mathsf{c}([\pi]) - \mathsf{c}([\pi'])| \le \rho_a([\pi], [\pi'])^{\alpha} \le C\,|\mathsf{c}([\pi]) - \mathsf{c}([\pi'])|, \quad [\pi], [\pi'] \in \partial^{g}\Gamma.$$

Work in progress

For infinite index set $I \subset \mathbb{N}$, let φ_i be a conformal IFS (cIFS) Ψ on compact set $X \in \mathbb{R}^d$. That is, each φ_i is a C^1-conformal injective contraction and there exists open bounded connected set $V \supset X$ such that φ_i can be extended to C^1-conformal injective contraction on V, $i \in I$ (cf. [MU96]). For $\omega \in I^*$ denote $R_\omega \coloneqq \sup_{v\in V} \|D\varphi_\omega(x)\|_{\mathrm{op}}$. Let Ψ have the bounded distortion property and satisfy $\lim_{i\in I} R_i = 0$. We can adapt the construction of an augmented tree for conformal case in an obvious way. Denote the limit set of Ψ by J.

Proposition

For fixed $\kappa > 0$, let $\Gamma = (\Gamma^*, \mathcal{E})$ be augmented tree for cIFS Ψ. Then the previous results are valid for Γ and limit set J.

- Γ is δ-hyperbolic graph and every horizontal geodesic is bounded in length by a constant γ.
- The geodesic boundary $\partial^{g}\Gamma$ is Hölder equivalent to the limit set J.

(Proof in progress.)

Open questions

- For IFS/ cIFS/ GDMS, does the open set condition guarantee the identity $\partial\Gamma = \partial^{g}\Gamma$?
- How does the concept of an augmented tree, in particular its hyperbolic and geodesic boundaries, relate to the concepts of limit set and (uniform) radial limit set of Kleinian groups?
- How do these concepts relate regarding the phenomenon of "dimension drop"?

References

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