The Sierpiński gasket as the Martin boundary of a non-isotropic Markov chain

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Abstract

In [5], Lau and Ngai, motivated by the work of Denker and Sato [1, 2], gave an example of an isotropic Markov chain on the set of finite words over a three letter alphabet, whose Martin boundary is homeomorphic to the Sierpiński gasket. Here, we show how these results can be extended to a class of non-isotropic Markov chains. This work is based on the recent article [3].

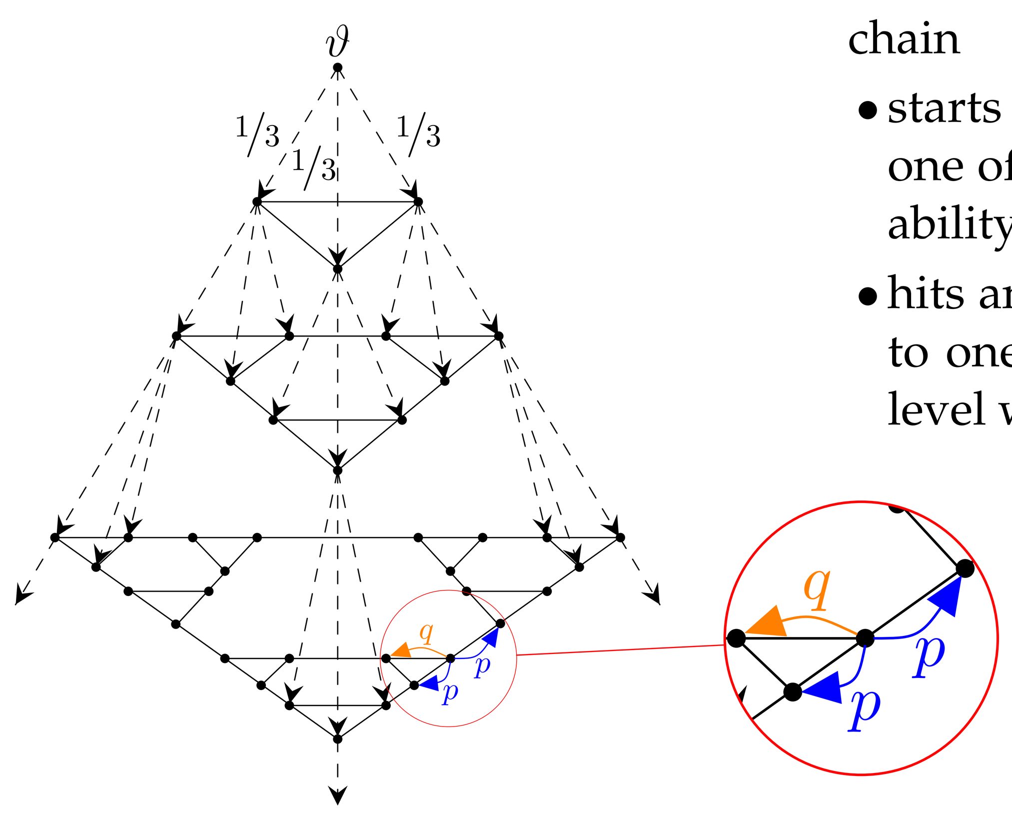
Setting and construction of the Markov chain

Let ϑ denote the empty word, $\Sigma^0 := \{\vartheta\}$ and $\Sigma := \{1,2,3\}$. Set $\Sigma^* := \bigcup_{n\in\mathbb{N}_0}\Sigma^n$, the set of all finite words over the alphabet Σ. Further, for $n \in \mathbb{N}$, let $V^n := \{1^n, 2^n, 3^n\}$ and set $\tilde{\Sigma}^n := \Sigma^n \setminus V^n$.

Let $p \in (0, 1/2)$ and set $q := 1 - 2p$. Define the transition matrix $P \colon \Sigma^* \times \Sigma^* \to [0,1]$ by

$$P(u,v) := \begin{cases} p & \text{if } u = \omega i j^{n-k} \in \tilde{\Sigma}^n \text{ with } i,j \in \Sigma \text{ distinct and } \omega \in \Sigma^{k-1} \\ & \text{and } v \in \Sigma^n \text{ with } v = \omega j i^{n-k} \text{ or } v = \omega i j^{n-k-1} i, \\ q & \text{if } u = \omega i j^{n-k} \in \tilde{\Sigma}^n \text{ with } i,j \in \Sigma \text{ distinct and } \omega \in \Sigma^{k-1} \\ & \text{and } v = \omega i j^{n-k-1} l \text{ for } l \in \Sigma \setminus \{i,j\}, \\ 1/3 & \text{if } u \in V^n \text{ and } v = ui \text{ for } i \in \Sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $(X_n)_{n\in\mathbb{N}_0}$ the Markov chain with origin ϑ, state space Σ^* and transition matrix P. Notice, if $p = 1/3$, then the above Markov chain coincides with the one in studied [5].



As the figure on the left illustrates, if the chain

- starts at a word in $\tilde{\Sigma}^n$, then it walks to one of its three neighbours with probability p or q;
- hits an element $u \in V^n$, then it moves to one of its descendants on the next level with probability $1/3$.

Hitting probabilities

Denote the probability, conditioned on starting at a state $x \in \Sigma^*$, to eventually arrive at a state $y \in \Sigma^*$ by

$$\rho(x,y) := \mathbb{P}(\exists\, k \in \mathbb{N}_0 : X_k = y \,|\, X_0 = x).$$

We are concerned with computing the probability to be absorbed by i^n, for $i \in \Sigma$, when starting at some $x \in \Sigma^n$. To this end, we define $\boldsymbol{\rho}\colon \Sigma^* \to [0,1]^3$ by

$$\boldsymbol{\rho}(x) := [\rho(x,1^n), \rho(x,2^n), \rho(x,3^n)].$$

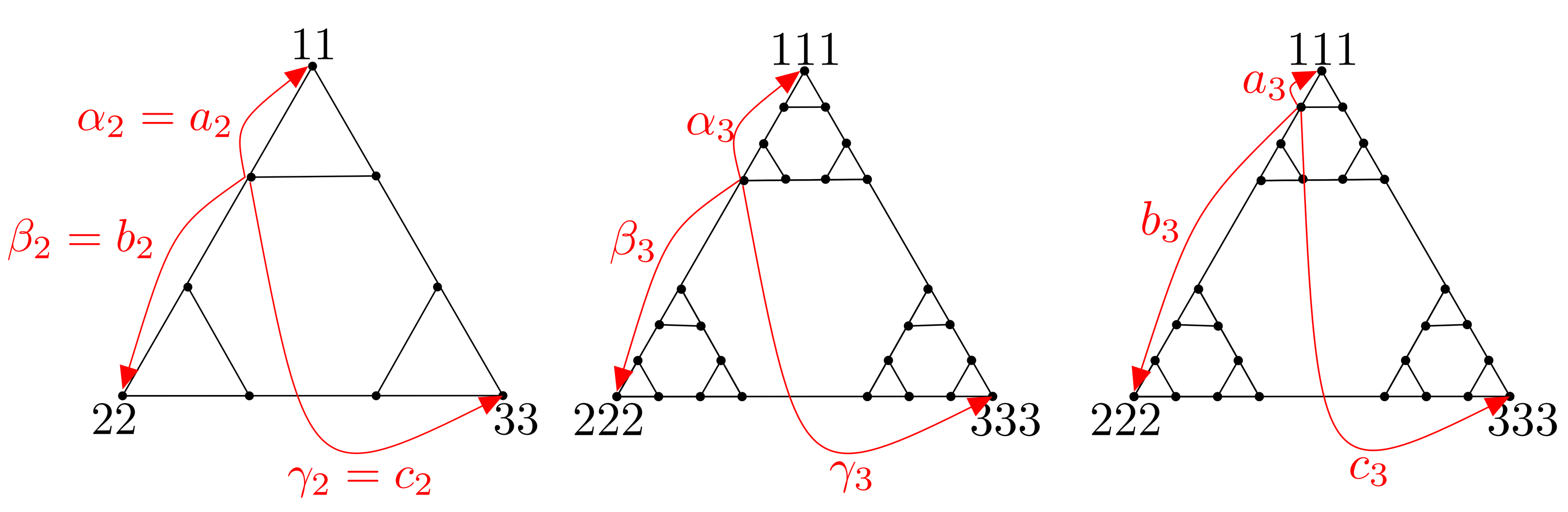
For $n \geq 2$ set

$$\begin{aligned} \alpha_n &:= \rho(12^{n-1}, 1^n), \\ \beta_n &:= \rho(12^{n-1}, 2^n), \\ \gamma_n &:= \rho(12^{n-1}, 3^n), \end{aligned} \qquad \text{and} \qquad \begin{aligned} a_n &:= \rho(1^{n-1}2, 1^n), \\ b_n &:= \rho(1^{n-1}2, 2^n), \\ c_n &:= \rho(1^{n-1}2, 3^n). \end{aligned}$$

Further, define

$$A_n^{(1)} := \begin{bmatrix} 1 & 0 & 0 \\ \alpha_n & \beta_n & \gamma_n \\ \alpha_n & \gamma_n & \beta_n \end{bmatrix}, \quad A_n^{(2)} := \begin{bmatrix} \beta_n & \alpha_n & \gamma_n \\ 0 & 1 & 0 \\ \gamma_n & \alpha_n & \beta_n \end{bmatrix} \quad \text{and} \quad A_n^{(3)} := \begin{bmatrix} \beta_n & \gamma_n & \alpha_n \\ \gamma_n & \beta_n & \alpha_n \\ 0 & 0 & 1 \end{bmatrix}.$$

The figure below shows the above hitting probabilities for $n = 2$ and $n = 3$.



The matrix $A_n^{(i)}$ contains the probabilities that the process, starting in one of the three vertices $i1^{n-1}$, $i2^{n-1}$ or $i3^{n-1}$, reaches V^n.

Denote the standard i-th row unit vector of $\mathbb{R}^3$ by $\boldsymbol{e}_i$ and let $x = i_1 \dots i_n \in \Sigma^n$. With the above at hand, we can express the hitting probability vector $\boldsymbol{\rho}(x)$ as a matrix product, namely,

$$\boldsymbol{\rho}(x) = \boldsymbol{e}_{i_n} A_2^{(i_{n-1})} \cdots A_n^{(i_1)}. \qquad (*)$$

Theorem 1: Limits of the hitting probabilities, [3]

$$\lim_{n\to\infty}(\alpha_n, \beta_n, \gamma_n) = (2/5, 2/5, 1/5) \quad \text{and} \quad \lim_{n\to\infty}(a_n, b_n, c_n) = (1, 0, 0).$$

The limits of these sequences are independent of the chosen parameter $p \in (0, 1/2)$ and are equal to the ones obtained in the isotropic case, namely when $p = 1/3$, see [5].

With these limits we can:

- prove that the random matrix product in $(*)$ converges;
- use a representation of the Martin kernel in terms of these hitting probabilities to extend the kernel to the set of infinite words over the alphabet Σ;
- prove that the Martin metric can also be extended to the set of infinite words over the alphabet Σ;
- find an analogue of the $(1/5)$-$(2/5)$-rule for the P-harmonic functions.

Main results

Theorem 2: Sierpiński gasket as Martin boundary, [3]

The Martin boundary of $(X_n)_{n\in\mathbb{N}_0}$ is homeomorphic to the Sierpiński gasket $\mathcal{K}$.

Theorem 3: Minimal Martin boundary, [3]

The minimal Martin boundary of $(X_n)_{n\in\mathbb{N}_0}$ is homeomorphic to the post critical set of $\mathcal{K}$.

Theorem 4: Space of P-harmonic functions, [3]

The P-harmonic functions on the Martin boundary coincide with the canonical harmonic functions of [4, 6]. Indeed, the space of P-harmonic functions on the Sierpińki gasket $\mathcal{K}$ is three-dimensional.

Future work

It would be of interest to investigate:

- what happens if we rotate the directions of the transition probabilities p and q;
- does the Martin boundary and P-harmonic functions change if the Markov chain is set up to prefer a clockwise or anti-clockwise direction in the subgraphs on each level;
- what happens if we choose different probabilities for each direction;
- can one modify the Markov chain such that the minimal Martin boundary is homeomorphic to a given Borel subset of $\mathcal{K}$?

Simulations indicate that for the first three points the same results as in Theorem 1 may hold.

References

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