Universität Bremen
Research Group
Dynamical Systems & Geometry

Random Cantor Set with dependence

Wafa Chaouch Ben Saad

(joint with Prof. Dr. Marc Keßeböhmer and Dr. Maxim Kirsebom)

University of Bremen

chaouch@uni-bremen.de

Abstract

Consider a random Cantor set that is generated by a binary tree-indexed family of random contractions. Without imposing the independence of the contractions we determine an almost sure upper bound of its Hausdorff dimension in terms of the random pressure function.

Introduction and Main Objective

Our purpose is to estimate an almost sure upper bound of the Hausdorff dimension of a random Cantor Set based on a tree indexed family and generated by some recursive contractions. Such a random fractal has been studied by K.Falconer [1, 2], S.Graf [5], R.D. Mauldin and S.C. Williams[4] and many others. The significant difference is that in their works the random fractals are generated by independent random contractions, while in our work we do not make any assumption on the linearity and the independence between the random contractions.

Construction and Notations

First an overview of the construction of this random Cantor Set. Formally the binary tree is the set:

$$T=\{\emptyset\}\cup\bigcup_{n=1}^{\infty}\{1,2\}^n$$

formed by the empty set and the words $u=u_1u_2...u_n$ in the alphabet $\{1,2\},\forall n\leq 1$ .We denote by $T_n=\{1,2\}^n$ the set of the words of length n. For $u\in T$ and $i\in\{1,2\}$ we denote by ui the concatenation of u and i.

We consider the interval $I=[0,1]$, and for any vertex u, we consider the closed and random interval I_u, which is a subset of $[0,1]$, and satisfies the following proprieties,

- If v is a child of u, then $I_v\subset I_u$.
- The two intervals I_{u1} and I_{u2} are disjoint.
- Finally, for every $u\in T$, the random contraction ratios satisfy

$$0<a\leq C_{ui}=\frac{|I_{ui}|}{|I_u|}\leq b<\frac{1}{2}\quad,\quad i=1,2,\text{ for some } a,b\in(0,1/2).$$

Define the random Cantor Set F by

$$F=\bigcap_{n\in\mathbb{N}}\bigcup_{u\in T_n}I_u.$$

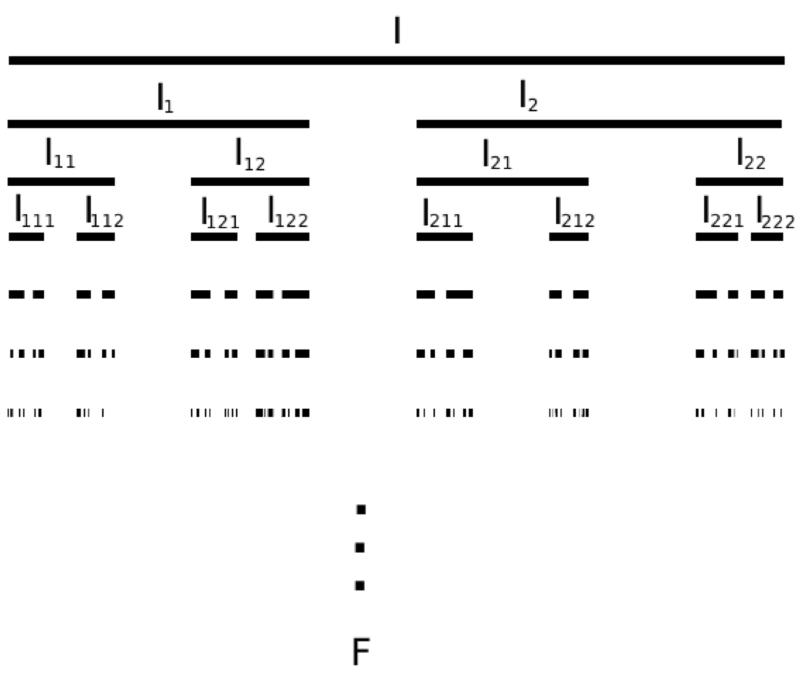

Figure 1: Random Cator Set

Let $(\Omega,\mathcal{F},P)$ be the probability space where the random variable C_u for $u\in T_n$ is defined. We impose the statistical self-similarity by requiring that for every $u\in T$, the vectors (C_{u1},C_{u2}) have the same distribution as the vector (C_1,C_2). Let $q\in\mathbb{R}$, we define the pressure function

$$P(q)=\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{E}\Big(\sum_{u\in T_n}|I_u|^q\Big)$$

Lemma 1: P is a **convex, continuous** and **strictly decreasing** function and thus there exists unique s_0, so that $P(s_0)=0$.

Main Result

Let s_0 be given as in lemma 1, then with probability 1, we have

$$\dim_H F\leq s_0.$$

Main idea of the proof: Prove that a.s.

$$\bar{P}(q)=\limsup_{n\to\infty}\frac{1}{n}\log\sum_{u\in T_n}|I_u|^q\leq P(q),\quad\forall q\in\mathbb{R}.$$

Examples for Equality and Strict Inequality

- **Equality** $\dim_H F_1=s_0$ **a.s.**

Example 1: Falconer [1, 2]

Falconer studied the same set but he required in addition that, for all $u\in T$, the vectors of contractions (C_{u1},C_{u2}) are independent (we call this Fractal F_1) and he gets the following result:

Theorem *With probability 1, the random Cantor Set F_1 has Hausdorff dimension $\dim_H F_1=s$, where s is the solution of the expectation equation*

$$\mathbb{E}(C_1^s+C_2^s)=1.$$

In other words s satisfies $P(s)=0$.

In this case, we get using the conditional expectation that our pressure function is exactly $P:q\in\mathbb{R}\longmapsto\log\mathbb{E}(C_1^q+C_2^q)$.

We notice that our result is consistent with Falconer's result.

- **Strict Inequality** $\dim_H F_2<s_0$ **a.s.**

Example 2: Hambly [6]

In this Example we impose in the construction of the random Cantor Set F_2 that $|I_u|=|I_v|$ when the words u and v have the same length,

and thus the C_u's are the same for all $u\in T_n$. In order to simplify the notation, we denote by $C_n:=C_u$ for any $u\in T_n$. Furthermore, we require that the random variables $\{C_n,n\in\mathbb{N}\}$ are ergodic and stationary (in particular independent) and we assume that $\mathbb{E}(\log C_1)<\infty$. In this case Hambly has the following result:

Theorem *With probability 1, the random set F_2 has Hausdorff dimension $\dim_H F_2=s$, where s is given by,*

$$s=-\frac{\log 2}{\mathbb{E}(\log C_1)}.$$

In other words s satisfies $\tilde{P}(s)=0$, where $\tilde{P}:q\in\mathbb{R}\longmapsto\limsup_{n\to\infty}n^{-1}\mathbb{E}\big(\log\sum_{u\in T_n}|I_u|^q\big)$.

In this example $\tilde{P}$ is exactly $\tilde{P}(q)=\log 2+q\mathbb{E}(\log C_1)$. Since, $\forall q\in\mathbb{R}$

$$\tilde{P}(q)=\limsup_{n\to\infty}\frac{1}{n}\mathbb{E}\Big(\log\sum_{u\in T_n}|I_u|^q\Big)\leq P(q)=\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{E}\Big(\sum_{u\in T_n}|I_u|^q\Big)$$

then, we can notice that the upper bound of the Hausdorff dimension of the random Cantor Set in this case is strictly less than s_0.

Remark: Analog results hold for random Multifactals.

Open Questions

- Are there further examples of random Cantor Sets which have a Hausdorff Dimension between the results of Example 1 and Example 2?
- Could we find an almost sure lower bound of the Hausdorff dimension of the random Cantor Set (when we do not make any assumption on the linearity and the independence between the random contractions)?
- What would the Hausdorff dimension of the Fractal in the second example be if we assume that the contractions are not linear? Could the t which satisfies $\tilde{P}(t)=0$ be the desired result?

References

[1] K.J. Falconer, Fractal Geometry. Mathematical Foundations and Applications.Wiley, 2nd Edition, 2003.

[2] K.J. Falconer, Random fractals. Math. Proc. Camb. Soc. (1986), 100, 559.

[3] K.J. Falconer, The multifractal spectrum of statistically self-similar measures. Journal of Theoretical Probability, Vol. 7. No. 3, (1994).

[4] R.D. Mauldin and S.C. Williams, Random recursive constructions: asymptotic geometric and topological properties. Trans. Amer. Math. (1986) Soc. 295 (1), 325-346.

[5] S. Graf, Random fractals, Rend. Istit. Mat. Univ. Trieste 23 (1991), no. 1, 81-144 (1993), School on Measure Theory and Real Analysis (Grado, 1991).

[6] B.M. Hambly, Brownian motion on a homogeneous random fractal, Probab. Theory Relat. Fields 94, 1 38 (1992).