Box dimension of graph of harmonic functions on the Sierpiński gasket

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Abstract

This poster presents, bounds for the box dimension of graph of harmonic function on the Sierpiński gasket. Also we get upper and lower bounds for the box dimension of graph of functions that belong to dom($\mathcal{E}$), that is, all finite energy functionals on the Sierpiński gasket. Further, we show the existence of fractal functions in the function space dom($\mathcal{E}$) with the help of fractal interpolation functions. Moreover, we provide bounds for the box dimension of some functions that belong to the family of continuous functions and arise as fractal interpolation functions.

Introduction

A range of fractals which we come across during studies occur as graph of functions. Certainly, many natural phenomena such as wind speed, solar radiation, population, stock market price etc. are plotted against time, that is, we capture them in graphs. The graph of functions and its box and Hausdorff dimension is of qualitative interest for many authors since past few decades.

Preliminaries

Definition 0.1 (Box Dimension). Let F be a nonempty subset of $\mathbb{R}^n$ and $N_\delta(F)$ denote the least number of sets of diameter less than or equal to δ which cover F. The lower box dimension (box-counting dimension) and the upper box dimension (box-counting dimension) of F is defined as

$$\underline{\dim}_B(F) = \liminf_{\delta\to 0^+} \frac{\log N_\delta(F)}{-\log\delta} \quad \text{and} \quad \overline{\dim}_B(F) = \limsup_{\delta\to 0^+} \frac{\log N_\delta(F)}{-\log\delta}$$

respectively. When these two values are equal, we call the common value as the box dimension of F.

Let $\mathcal{S}_0 = \{q_1, q_2, q_3\}$ be three points on $\mathbb{R}^2$ equidistant from each other. Let $L_i(x) = \frac{1}{2}(x - q_i) + q_i$ for $i = 1, 2, 3$ and $L : \mathcal{B}(\mathbb{R}^2) \to \mathcal{B}(\mathbb{R}^2)$ defined as $L(A) = \cup_{i=1}^3 L_i(A)$. It is well known that L has a unique fixed point $\mathcal{S}$, which is called the Sierpiński gasket. We define energy functional on the space of continuous functions on the Sierpiński gasket($\mathcal{C}(\mathcal{S})$) as follows. The m^{th} level Sierpiński gasket is $\mathcal{S}^{(m)} \coloneqq \cup_{j=0}^m L^j(\mathcal{S}_0)$. If x and y belongs to same cell of $\mathcal{S}^{(m)}$ we denote it by $x \sim_m y$. We define the m^{th} level crude energy as

$$E^{(m)}(u) = \sum_{x\sim_m y} |u(x) - u(y)|^2$$

and the m^{th} level renormalized energy is given by $\mathcal{E}^{(m)}(u) = \left(\frac{5}{3}\right)^m E^{(m)}(u)$, where $\frac{5}{3}$ is the unique renormalizing factor. Now we can observe that $\mathcal{E}^{(m)}(u)$ is a monotonically increasing function of m because of renormalization. So we define the energy function as

$$\mathcal{E}(u) = \lim_{m\to\infty} \mathcal{E}^{(m)}(u)$$

which exists for all u as an extended real number. Now we define dom($\mathcal{E}$) as the space of continuous functions u satisfying $\mathcal{E}(u) < \infty$. It has been proved that dom($\mathcal{E}$) modulo constant functions forms a Banach space endowed with the norm $\|\cdot\|_{\mathcal{E}}$ defined as $\|u\|_{\mathcal{E}} = \sqrt{\mathcal{E}(u)}$. The space dom$_0(\mathcal{E})$ is a subspace of dom($\mathcal{E}$) containing all functions which vanish at boundary of the Sierpiński gasket.

Definition 0.2 (Harmonic function). A function $f : \mathcal{S} \to \mathbb{R}$ is said to be a harmonic function if $\mathcal{E}^{(m+1)}(f) = \mathcal{E}^{(m)}(f)$ for every $m \geq 0$.

A harmonic function satisfies the following rule known as "$\frac{1}{5} - \frac{2}{5}$" rule

$$f(q_{\omega ij}) = \frac{2}{5}h(q_{\omega i}) + \frac{2}{5}h(q_{\omega j}) + \frac{1}{5}h(q_{\omega k})$$

where $\omega \in \Sigma^*$ and $\{i, j, k\}$ are permutation of $\{1, 2, 3\}$. We define $\Sigma^* \coloneqq \cup_{m\geq 0}\Sigma^{(m)}$ and $\Sigma^{(m)}$ is the collection of all words of length m which are possible combinations of symbols 1,2 and 3. We define, $q_{\omega i} \coloneqq L_\omega(q_i)$ for $\omega \in \Sigma^*$ and $i \in \{1, 2, 3\}$.

Fractal operator on dom($\mathcal{E}$) and its properties

Now the question arises : Is there any fractal function in the space dom($\mathcal{E}$)? And the answer is affirmative.

Construction of α-fractal functions on the Sierpiński gasket

Let f be a continuous function on the Sierpiński gasket and $n \in \mathbb{N}$ be a fixed number. Suppose the interpolation points are $\{(q_\omega, f(q_\omega)) : \omega \in \Sigma^{(n)}\}$. Then for fixed $\alpha = \{\alpha_\omega \in (-1, 1) : \omega \in \Sigma^{(n)}\}$ we will construct an IFS such that the attractor is graph of a function and passes through the above interpolation points. We define a function $L_\omega : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$L_\omega \coloneqq L_{w_1} \circ L_{w_2} \circ L_{w_3} \circ \ldots \circ L_{w_n} \text{ where } \omega \in \Sigma^{(n)} \tag{1}$$

The function L_ω satisfies following conditions

$$L_\omega(q_1) = q_{\omega 1}, L_\omega(q_2) = q_{\omega 2}, L_\omega(q_3) = q_{\omega 3} \quad \text{and} \quad \|L_\omega(c) - L_\omega(d)\| \leq \frac{1}{2^{|\omega|}}\|c - d\|.$$

Further, we define a real valued continuous function $F_\omega : \mathcal{S} \times \mathbb{R} \to \mathbb{R}$ by

$$F_\omega(\mathbf{x}, y) = \alpha_\omega y + f(L_\omega(\mathbf{x})) - \alpha_\omega\, b(\mathbf{x}) \tag{2}$$

where b is a continuous function on the Sierpiński gasket satisfying conditions $b(q_1) = f(q_1), b(q_2) = f(q_2)$ and $b(q_3) = f(q_3)$. The function F_ω satisfies following conditions

$$F_{3\tilde{\omega}}(q_1, y_1) = F_{1\tilde{\omega}}(q_3, y_3), F_{2\tilde{\omega}}(q_3, y_3) = F_{3\tilde{\omega}}(q_2, y_2) \text{ and } F_{2\tilde{\omega}}(q_1, y_1) = F_{1\tilde{\omega}}(q_2, y_2)$$

for each $\tilde{\omega} \in \Sigma^{(n-1)}$ and $\|F_\omega(c, d_1) - F_\omega(c, d_2)\| \leq |\alpha_\omega|\|d_1 - d_2\|$. Now we define the IFS using the above functions

$$\text{IFS}\{K; \mathbf{f}_\omega : \omega \in \Sigma^{(n)}\} \text{ where } \mathbf{f}_\omega(\mathbf{x}, y) = (L_\omega(\mathbf{x}), F_\omega(\mathbf{x}, y)). \tag{3}$$

This is a contractive IFS. Hence has a unique attractor, say G. The function corresponding to graph G is named f^α which passes through the interpolation points $\{(q_\omega, f(q_\omega)) : \omega \in \Sigma^{(n)}\}$. The above function f^α satisfies the functional equation

$$f^\alpha(\mathbf{x}) = f(\mathbf{x}) + \alpha_\omega(f^\alpha - b) \circ L_\omega^{-1}(\mathbf{x}), \ \forall\ \mathbf{x} \in L_\omega(\mathcal{S}). \tag{4}$$

In the above construction if we take $b = T(f)$ where $T : \mathcal{C}(\mathcal{S}) \to \mathcal{C}(\mathcal{S})$ is a bounded linear operator then we define

Definition 0.3 (α-fractal operator). We define the α-fractal operator $\mathcal{F}^\alpha = \mathcal{F}^\alpha_{n,T}$ on $\mathcal{C}(\mathcal{S})$ with respect to fixed n, α and T as

$$\mathcal{F}^\alpha(f) = f^\alpha$$

where f^α is the α-fractal function corresponding to f.

Theorem 0.4. *Let f be a function in dom($\mathcal{E}$) and f^α be the α-fractal function corresponding to f. Then the function f^α belongs to dom($\mathcal{E}$) if $\|\alpha\|_\infty \leq \frac{1}{\sqrt{3\times 5^n}}$ where n is the fixed number associated to interpolation points, that is, $\{(q_\omega, f(q_\omega)) : \omega \in \Sigma^{(n)}\}$.*

Corollary 0.5. *If T is a linear bounded operator with respect to uniform norm and $\|\alpha\|_\infty \leq \frac{1}{\sqrt{3\times 5^n}}$ then $\mathcal{F}^\alpha$: dom($\mathcal{E}$) $\to$ dom($\mathcal{E}$) is linear and bounded with respect to $\|\cdot\|_{\mathcal{E}}$ norm and $\|\mathcal{F}^\alpha\|_{\mathcal{E}} \leq \sqrt{\frac{(3+5^n 3\|\alpha\|^2\|T\|)}{(1-5^n 3\|\alpha\|^2)}}$.*

Corollary 0.6. *If $f \in$ dom($\mathcal{E}$) then $\|f^\alpha - f\|_{\mathcal{E}} \leq \|\alpha\|_\infty^2 3^n \|f^\alpha - Tf\|_{\mathcal{E}}$.*

Box dimension of graph of fractal functions

In this section we will obtain upper and lower bounds for the box dimension of graph of fractal functions. We fix $q_1 = (0, 0), q_2 = (1, 0), q_3 = (\frac{1}{2}, \frac{\sqrt{3}}{2}), n = 1$ and interpolation points are

$$\{(q_1, f(q_1)), (q_2, f(q_2)), (q_3, f(q_3)), (q_{12}, f(q_{12})), (q_{13}, f(q_{13})), (q_{23}, f(q_{23}))\}.$$

We construct the IFS $\{K, \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$. We define $L_i(\mathbf{x}) = \frac{1}{2}(\mathbf{x} - q_i) + q_i$ and $F_i(\mathbf{x}, y) = \alpha_i y + f(L_i(\mathbf{x})) - \alpha_i b(\mathbf{x})$. Here $b \in \mathcal{C}(\mathcal{S})$ and satisfies $b(q_i) = f(q_i)$ for all $i = 1, 2, 3$. So, $\mathbf{f}_i(\mathbf{x}, y) = (L_i(\mathbf{x}), F_i(\mathbf{x}, y))$ for all $i = 1, 2, 3$. The fixed point of this IFS gives graph of a function. We try to estimate the box dimension of this graph under certain conditions.

Theorem 0.7. *Let f and b be Hölder continuous functions with exponent η_1, η_2 respectively and the interpolation points are not coplanar. Let f^αbe the α-fractal function corresponding to f and $G = \{(\mathbf{x}, f^\alpha(\mathbf{x})) : \mathbf{x} \in \mathcal{S}\}$ be the graph of f^α. Let $\psi = \sum_{i=1}^3 \alpha_i$ and $\eta = \min\{\eta_1, \eta_2\}$. Then the box dimension of G has following bounds :*
(I) If $\frac{\psi 2^\eta}{3} \leq 1$, then $\frac{\log 3}{\log 2} \leq \dim_B(G) \leq 1 - \eta + \frac{\log 3}{\log 2}$.
(II)If $\frac{\psi 2^\eta}{3} > 1$, then $\frac{\log 3}{\log 2} \leq \dim_B(G) \leq 1 + \frac{\log\psi}{\log 2}$.

Box dimension of graph of Harmonic functions

Lemma 0.8. *Let h be a harmonic function satisfying $|h(\mathbf{x}) - h(\mathbf{y})| \leq \left(\frac{6}{5}\right)^m \|\mathbf{x} - \mathbf{y}\|$ for each pair of $\mathbf{x}$ and $\mathbf{y}$ in the same cell in the level $\mathcal{S}^{(m)}$. Then $|h(\tilde{\mathbf{x}}) - h(\tilde{\mathbf{y}})| \leq \left(\frac{6}{5}\right)^{(m+1)} \|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}\|$ for each $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ that belong to same cell in the level $\mathcal{S}^{(m+1)}$ and $\tilde{\mathbf{x}}, \tilde{\mathbf{y}}$ belong to sub cell of the cell containing $\mathbf{x}$ and $\mathbf{y}$.*

Theorem 0.9. *Let h be a harmonic function. Then the box dimension of graph of h is less than or equal to $\frac{\log(18/5)}{\log 2} \approx 1.8479$.*

Proof. Let h be a harmonic function and G_h be the graph of h. We observe that using Lemma 0.8 repeatedly, we get

$$|h(\mathbf{x}) - h(\mathbf{y})| \leq \left(\frac{6}{5}\right)^m \|h\|_{\mathcal{E}}\|\mathbf{x} - \mathbf{y}\|$$

whenever $\mathbf{x}$ and $\mathbf{y}$ belongs to same cell of the $\mathcal{S}^{(m)}$.Hence, we get upper bound for the box counting dimension is

$$\dim_B(G_h) \leq \overline{\lim}_{\delta\to 0}\frac{\log N_\delta(G_h)}{-\log\delta} \leq \overline{\lim}_{m\to\infty}\frac{\log 3^m\left(\left(\frac{6}{5}\right)^m\|h\|_{\mathcal{E}} + 2\right)}{-\log 2^{-m}} = \frac{\log(18/5)}{\log 2} \approx 1.8479.$$

□

Corollary 0.10. *Let h be a piecewise harmonic function on the Sierpiński gasket. Then the box counting dimension is less than or equal to$\frac{\log(18/5)}{\log 2} \approx 1.8479$.*

Lemma 0.11. *Let u be any function in dom($\mathcal{E}$) then we have $|u(\mathbf{x}) - u(\mathbf{y})| \leq \left(\frac{3}{5}\right)^{m/2}\sqrt{\mathcal{E}(u)}$ whenever x and y belong to the same cell of $\mathcal{S}^{(m)}$.*

Lemma 0.12. *Let $F \subset \mathbb{R}^n$ and suppose that $f : F \to \mathbb{R}^m$ is a Lipschitz transformation, that is, there exists $c \geq 0$ such that $|f(x) - f(y)| \leq c|x - y|$ for every $x, y \in F$. Then $\underline{\dim}_B f(F) \leq \underline{\dim}_B F$ and $\overline{\dim}_B f(F) \leq \overline{\dim}_B F$.*

Theorem 0.13. *Let f be any function in dom($\mathcal{E}$). Then the box dimension of graph of f has lower bound $\frac{\log 3}{\log 2}$ and upper bound $\frac{\log(108/5)}{2\log 2} \approx 2.21648$.*

Proof. Let f be any function in dom($\mathcal{E}$) and G_f be the graph of f. Define $P : G_f = (\mathbf{x}, f(\mathbf{x})) \to \mathbb{R}^2$ as $P(\mathbf{x}, f(\mathbf{x})) = \mathbf{x}$. Clearly, P is a Lipschitz map and $P(G_f) = \mathcal{S}$. Hence, by Lemma 0.12 we have $\dim_B P(G_f) \leq \dim_B G_f$ which is same as $\underline{\dim}_B\mathcal{S} \leq \underline{\dim}_B G_f$. This implies $\dim_B G_f \geq \frac{\log 3}{\log 2}$.

From Lemma 0.11 we have $|f(\mathbf{x}) - f(\mathbf{y})| \leq \left(\frac{3}{5}\right)^{m/2}\sqrt{\mathcal{E}(u)}$ whenever $\mathbf{x}, \mathbf{y}$ belong to same cell of $\mathcal{S}^{(m)}$ and it holds true for every $m \geq 0$. Hence to cover the graph of each cell of $\mathcal{S}^{(m)}$ with cube of side length 2^{-m} we need $\left(\frac{12}{5}\right)^{m/2}\sqrt{\mathcal{E}(u)} + 2$ many cubes. To cover graph of $\mathcal{S}^{(m)}$ we need at most $3^m\left(\left(\frac{12}{5}\right)^{m/2}\sqrt{\mathcal{E}(u)} + 2\right)$ many cubes. Hence, we get upper bound for box counting dimension as

$$\dim_B(G_f) = \overline{\lim}_{\delta\to 0}\frac{\log\mathcal{N}_\delta(G_f)}{-\log\delta} \leq \overline{\lim}_{m\to\infty}\frac{\log\left(\left(\frac{108}{5}\right)^{m/2}\sqrt{\mathcal{E}(u)} + 2\times 3^m\right)}{-\log 2^{-m}} = \frac{\log(108/5)}{2\log 2} \approx 2.21648.$$

□

Remark 0.14. Any constant function $f : \mathcal{S} \to \mathbb{R}$ is harmonic, $f \in$ dom($\mathcal{E}$) and

$$\dim_H(G_f) = \underline{\dim}_B(G_f) = \overline{\dim}_B(G_f) = \dim_B(G_f) = \frac{\log 3}{\log 2}.$$

Hence, we can say that the lower bound in theorem 0.13 is attained.

References

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