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The dimensions of inhomogeneous self-affine sets

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Abstract

We prove that the upper box dimension of an inhomogeneous self-affine set is bounded above by the maximum of the affinity dimension and the dimension of the condensation set. In addition, we determine sufficient conditions for this upper bound to be attained, which, in part, constitutes an exploration of the capacity for the condensation set to mitigate dimension drop between the affinity dimension and the corresponding homogeneous attractor. Our work improves and unifies previous results on general inhomogeneous attractors, low-dimensional affine systems, and inhomogeneous self-affine carpets, while providing inhomogeneous analogues of Falconer's seminal results on homogeneous self-affine sets.

Setting the scene

An *iterated function system* (IFS) is a finite collection $\{S_i\}_{i=1}^N$ of contracting maps on a compact metric space (X, d). It follows from an elegant application of Banach's contraction mapping theorem that for each compact set C, there exists a unique non-empty compact set F_C such that

$$F_C = \bigcup_{i=1}^{N} S_i(F_C) \cup C,$$

called the *inhomogeneous* attractor, or inhomogeneous set, with *condensation set* C. For example, see Figure 1. We say the attractor is an inhomogeneous **self-affine set** if the IFS consists of affine transformations.

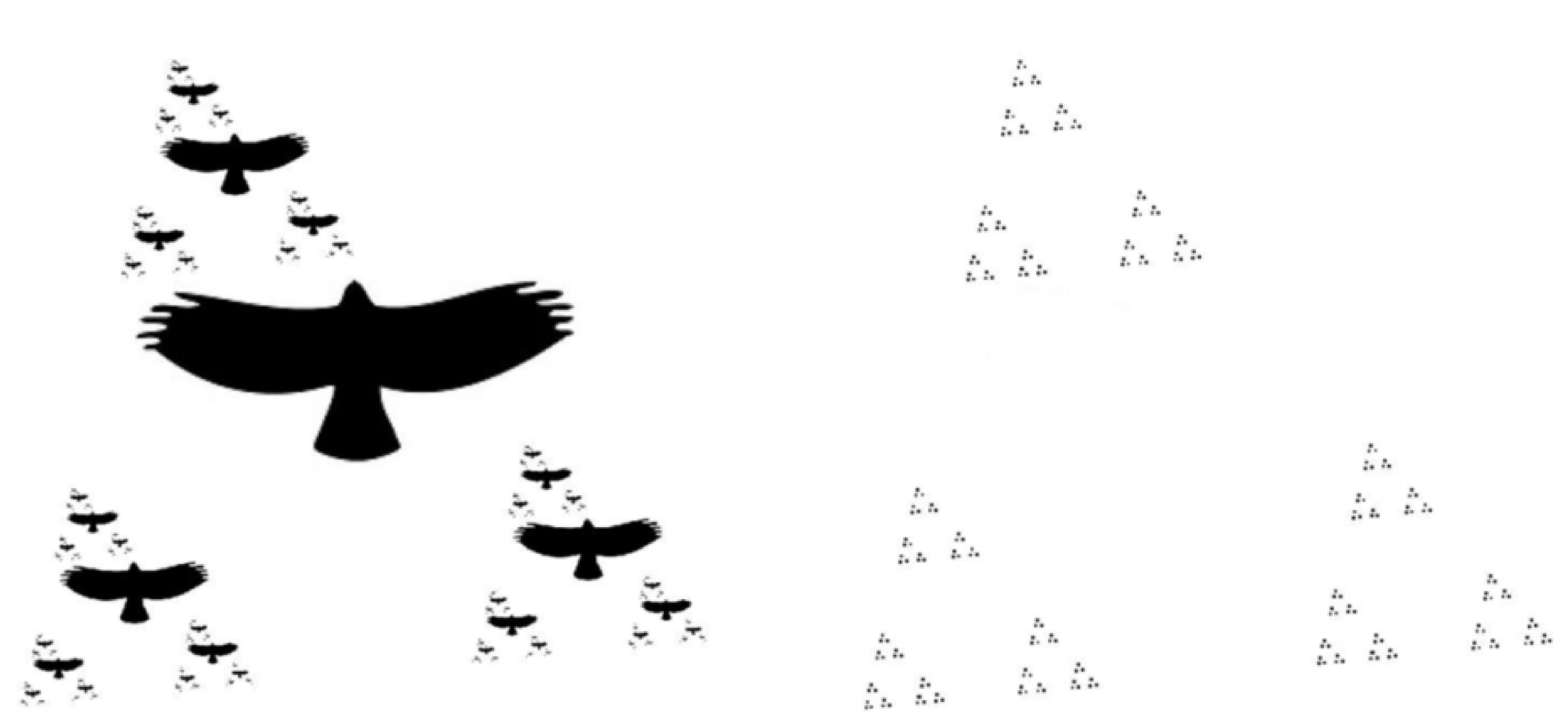

Figure 1: A flock of birds: a simple inhomogeneous set with condensation set the central bird (left), and the related homogeneous attractor corresponding to $C = \emptyset$ (right) [4].

A snippet of history

- A central question in this field has been to determine in what situations
$$\overline{\dim}_{\mathrm{B}} F_C = \max\{\overline{\dim}_{\mathrm{B}} F_\emptyset, \overline{\dim}_{\mathrm{B}} C\}.$$
- One approach is to establish bounds of the form
$$\max\{\overline{\dim}_{\mathrm{B}} F_\emptyset, \overline{\dim}_{\mathrm{B}} C\} \leq \overline{\dim}_{\mathrm{B}} F_C \leq \max\left\{s, \overline{\dim}_{\mathrm{B}} C\right\}, \qquad (1)$$
where s is some estimate of $\overline{\dim}_{\mathrm{B}} F_\emptyset$.
- This reduces the problem to understanding when $\overline{\dim}_{\mathrm{B}} F_\emptyset = s$, which in many contexts is well understood.

The natural question

For *self-similar* sets, Fraser established (1) with s equal to similarity dimension [4], while Burrell introduced upper Lipschitz dimension to obtain (1) for arbitrary bi-Lipschitz IFSs [1], yielding sharp results for self-conformal sets.

Is affinity dimension the optimum estimate and correct choice of s for inhomogeneous self-affine sets?

Results

The main result of [2] answers this question affirmatively, and may be considered an inhomogeneous analogue of Falconer's seminal result on homogeneous self-affine sets [3], which establishes $\overline{\dim}_{\mathrm{B}} F_\emptyset \leq s$.

Theorem 1

Let $F_C \subset \mathbb{R}^n$ be an inhomogeneous self-affine set with compact condensation set $C \subset \mathbb{R}^n$. We have

$$\max\{\overline{\dim}_{\mathrm{B}} F_\emptyset, \overline{\dim}_{\mathrm{B}} C\} \leq \overline{\dim}_{\mathrm{B}} F_C \leq \max\left\{s, \overline{\dim}_{\mathrm{B}} C\right\},$$

where s is the affinity dimension associated with the underlying IFS.

In addition, we found that C may mitigate dimension drop between the affinity dimension and the homogeneous attractor. That is, there exist situations when $s > \overline{\dim}_{\mathrm{B}} F_\emptyset$ but

$$\overline{\dim}_{\mathrm{B}} F_C = \max\left\{s, \overline{\dim}_{\mathrm{B}} C\right\}.$$

An example of this is given in Figure 2, where $\overline{\dim}_{\mathrm{B}} F_\emptyset = 0$ but it may easily be shown that $\overline{\dim}_{\mathrm{B}} F_C = s > 1$.

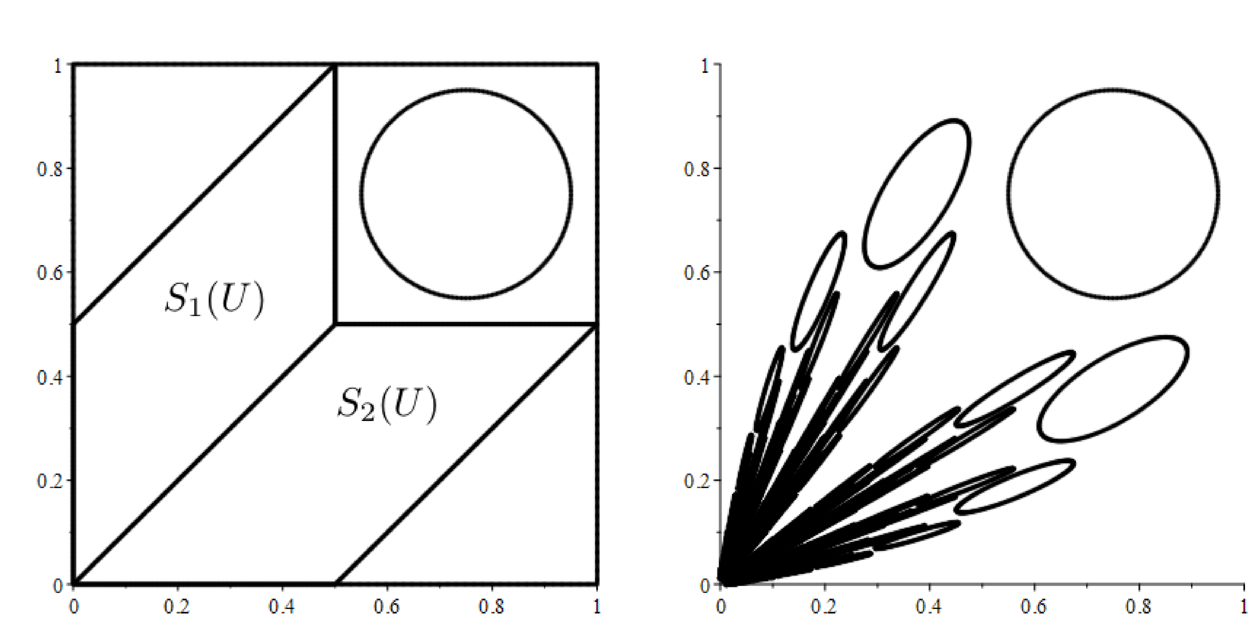

Figure 2: A bouquet of ovals: an inhomogeneous self-affine set with C mitigating dimension drop.

We have determined various sufficient conditions for this to occur. For example, in the plane, if C has dimension greater than 1 and is not contained in a line. This extends to higher dimensions if C is in some sense robust under projections.

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